Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

G.K.V. LIB, Hardwar

Bull. Cal. Math. Soc.

Vol. 86 No. 3 5 6 1999

Akshara Samaj Foundation Chennai and Kangri Collection

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri
130271

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

BULLETIN OF THE CALCUTTA MATHEMATICAL SOCIETY

130271

Vol. 86

130271

Gurukul Kangri Vishwavidyalaya
HARIDWAR

CALCUTTA MATHEMATICAL SOCIETY 1908

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

BULLETIN OF CALCUTTA MATHEMATICAL SOCIETY

Editorial Secretary: C. G. Chakrabarti, AE 374, Sector.I., Salt Lake City, Calcutta 700 064

BOARD OF EDITORS

P. Erdös (HUNGARY). (Discrete Mathematics, Mathematical Analysis)
F. Harary (U.S.A.), (Discrete Mathematics)
N. Jacobson (U.S.A.), (Abstract Algebra)
M. Dutta (INDIA), (Abstract Mathematics, Mathematical Sciences)
S. Dutta Majumdar (INDIA), (Quantum Mechanics)
N. D. Sen Gupta (INDIA), (Quantum Mechanics)
P. C. Vaidya (INDIA), (Relativity, Manifold Theory)
A. K. Choudhury (INDIA), (Control Theory, Operation Research)
Th: M. Rassias (GREECE) (Mathematical Analysis)
S. M. Shah (U.S.A.), (Mathematical Analysis)
H. M. Srivastava (CANADA), (Mathematical Analysis)
G. Jumarie (CANADA), (Cybernetics and Systems)
B. H. Lavenda (ITALY), (Thermodynamics, Statistical Physics)
H. I. Andersson (NORWAY), (Fluid Mechanics)
L. Debnath (U.S.A.). (Mathematical Analysis, Continuum Mechanics)
B. D. Sharma (U.S.A.), (Information and Coding Theory)
J. Gopala Krishnan (INDIA), (Mathematical Analysis)
K. Ramachandra (INDIA), (Theory of Numbers)
S. C. Dasgupta (INDIA), (Continuum Mechanics)
J. K. Ghosh (INDIA), (Mathematical Statistics)
A. S. Gupta (INDIA), (Fluid Mechanics)
K. R. Unni (INDIA), (Mathematical Analysis)
Karmeshu (INDIA), (Stochastic Processes, Socio Political System)
M. R. Gupta (INDIA), (Plasma Physics, Dynamical System)
S. K. Chatterjea (INDIA), (Mathematical Analysis)
T. K. Mukherjee (INDIA), (Algebra, Mathematical Analysis)
B. N. Mandal (INDIA), (Continuum Mechanics, Operation Research)
C. G. Chakrabarti (INDIA), (Statistical Physics, Mathematical Biology)

Benefactors : Calcutta University (India); University of Central Florida (U.S.A)

Communications : For publication in the 'Bulletin of Calcutta Mathematical Society' Papers should be written according to the format of Bulletin (i.e. Notes for authors) and are to be sent to the Editorial Secretary or to any member of the Editorial Board.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Declaration No. 111/88

Digitized by Arya Samaj Foundation Chennai and eGangotri

BULLETIN OF THE CALCUTTA MATHEMATICAL SOCIETY

Volume 86 Number 3 June 1994

CONTENTS

M. BELTAGY
On the Geometry of Specific Types of Points 193
SHAHWAR F. RAGAB
Infinite Series Involving Bessel Functions of the same orders 201
A.M.EL.-NAGGAR, A.M. ABD-ALLA AND S.M. AHMED
The Stresses in a Rotation of a Non-homogeneous Composite Cylinder of Isotropic Material 205
RAM HIT OJHA AND BHAGWAT PRASAD
On a Semi-Symmetric Non-Metric Connexion in Almost Grayan Manifolds 213
A. K. SINGH AND K. C. PETWAL
On Decompositon of Curvature Tensor Fields in Tachibana Recurrent Spaces 221
M. H. SALLAM
Generalised Simple Eigenvalues and Bifurcation for a Linked Multiparameter Eigenvalue Problem 227
RATNA CHAKRAVARTY (HAZRA) AND D. K. BASU
An Extension of a Bilateral Generating Function of Modified Jacobi Polynomial 233
GARGI CHAKRABORTY
Existence Theorems, Uniqueness for General Formulation of the Minimum Effort Problems in Locally Convex Linear Topological Spaces 237
A.M.EL-NAGGAR, A. M. ABD-ALLA AND S. M. AHEMD
Rayleigh Waves in a Magnetoelastic Initially Stressed Conducting Medium with the Gravity Field 243
MIGUEL CALDAS CUEVA
Further Results on Non-Archimedean σ Barrelled in Locally Convex Spaces 249
N. PARHI AND P. DAS
Oscillatoy and Asymptotic Behaviour of a Class of Non-Linear Functional Differential Equations of Third Order 253
ADEL M. ZAKI
The Pure State Spaces of Certain C*-Algebras 267
ABDULLA S. BIN SHAHNA
Fuzzy Connectedness and Weaker Forms of Fuzzy Continuity 271
K. V. V. SESHAGIRI RAO
Non-Oscillation for a Class of Forced Second Order Non-Linear Differential Equations with Damping 275
SHABNAM KAHN, D. P. SHUKLA AND N. P. S. BAWA
On Computation of Triangulation with Continuous Refinement of Grid Size 279

Per volume : (For 1994) Rs. 325.00 (India), or $ 125.00 (Foreign)
Subscription Rate per back volume : Rs. 550.00 (India) or $ 125.00
PRINTED IN INDIA
PRINTED AND PUBLISHED BY DR. B. N. MANDAL ON BEHALF OF CALCUTTA MATHEMATICAL SOCIETY
AE 374, SECTOR I, SALT LAKE CITY, CALCUTTA 700 064, INDIA AND PRINTED AT BOOKS & ALLIED (P) LTD., 8/1 CHINTAMONI DAS LANE, CALCUTTA 700 009
EDITORIAL SECRETARY : DR. C. G. CHAKRABARTI.

130271
CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

BULLETIN of
Calcutta Mathematical Society

Bi-monthly
internationally reputed
Journal of Mathematics and
Mathematical Sciences.
85 years of continuous publications.
Circulated all over the world.

NEWS BULLETIN of
Calcutta Mathematical Society

A sister Journal of Bulletin of Calcutta Mathematical Society publishes semi-popular (without complicated mathematics) review articles, articles of pedagogic interest, book reviews, Educational news and academic activities of Calcutta Mathematical Society.

Communications should be made with:

Dr. C. G. Chakrabarti
Editorial Secretary.
AE 374, Sector I, Salt Lake City
Calcutta 700 064 (INDIA)

Review Bulletin of The Calcutta Mathematical Society

Bi-annual Review Journal in addition to its Bulletin and News Bulletin.
The good review articles, the lectures delivered in the Society
which are not out and out research articles nor popular articles
are published in this Journal and circulated internationally.

The Review Bulletin is published Bi-annually, in June and December.

Suscription rate per volume for 1993 Rs. 200.00 (INDIA) or $ 75.00 (FOREIGN).

In sending manuscript of papers, please supply the following information
(i) Major field(s) (ii) AMS subject classification.

The Bulletin is published Bi-monthly, in February, April, June, August, October, December.

Subscription rate per volume for 1994 Rs. 325.00 (INDIA) or $125.00 (FOREIGN)

Subscription rate per back volume Rs. 550.00 or $125.00 (FOREIGN)

The News Bulletin is published monthly.

Subscription rate per volume for 1994 Rs. 36.00 (INDIA) or $36.00 (FOREIGN)

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Declaration No. 111/88

BULLETIN OF THE CALCUTTA MATHEMATICAL SOCIETY

Volume 86 Number 5 October 1994

CONTENTS

N. K. GHOSH
On Axisymmetric Waves in Water with an Inertial Surface 373

H. ÇAKALLI AND G. CANAK
s—Absolute Almost Convergent Sequence II 379

M. BHATTACHARJYA AND J. RATH
A Generalisation of The Generating Functions of Boas and Buck 387

S. K. SEN AND M. N. MUKHERJEE
θ-Perfect Functions Between Bitopological Spaces 395

B. B. SINHA AND R. N. YADAV
Semi-Invariant Submanifolds Of a Kenmotsu Manifold 405

SHANTESHWAR PRASAD AND RAM HIT OJHA
On M-Manifolds 413

M. I. M. AL-ALI AND NGUYEN TO NHU
Roberts' Example And The AR-Property of Convex Sets In Non-Locally Convex Linear Metric Spaces 419

SHIH-CHUAN CHENG AND JOHN N. MORDESON
Fuzzy Linear Operators and Fuzzy Normed Linear Spaces 429

T. K. DUTTA AND N. C. ADHIKARI
On Prime Radical Of Γ-Semigroup 437

S. K. BHAMBRI, RAJESH KUMAR AND PRATIBHA KUMAR
On Fuzzy Primary Submodules 445

YU BINGJUN
The Biorder And Sandwich Sets On A Semigroup With Idempotents 453

GOPI AHUJA
On The Product Of An Arbitrary Function And The Fractional Derivative Of Dirac-Delta Distribution-II 463

A. S. MASHHOUR, F. M. ZEYADA AND A. H. ISMAIL
On Good Extensions of Some Topological Covering Properties 469

130271

Per volume : (For 1994) Rs. 325/– (India), or $ 125.00 (Foreign)
Subscription Rate per back volume : Rs. 550/– (India) or $ 125.00

PRINTED IN INDIA
PRINTED AND PUBLISHED BY DR. R. N. SEN ON BEHALF OF
CALCUTTA MATHEMATICAL SOCIETY
AE 374, SECTOR I, SALT LAKE CITY, CALCUTTA-700 064, INDIA AND PRINTED AT
EASTERN PRINTERS, 71 HARI GHOSH STREET, CALCUTTA-700 006
EDITORIAL SECRETARY : DR. C. G. CHAKRABARTI.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bulletin of The Calcutta Mathematical Society

Bi-monthly internationally reputed Journal of Mathematics and Mathematical Sciences. 85 years of continuous publications. Circulated all over the world.

News Bulletin of Calcutta Mathematical Society

A sister Journal of Bulletin of Calcutta Mathematical Society published semipopular (without complicated mathematics) review articles, articles of pedagogic interest, book reviews, Educational news and academic activities of Calcutta Mathematical Society.

Communications should be made to :

Dr. C. G. Chakrabarti,
AE-374, Sector-1, Salt Lake City,
Calcutta-700 064 (INDIA)

Review Bulletin of The Calcutta Mathematical Society

Bi-annual Review Journal in addition to its Bulletin and News Bulletin. The Good review articles, the lectures delivered in the Society which are not out and out research articles nor popular articles are published in this Journal and circulated internationally.

The Review Bulletin is published Bi-annually, in June and December. Subscription rate per volume for 1993 Rs. 200.00 (INDIA) for $ 75.00 (foreign).

In sending manuscript of papers, please supply the following information (i) Major field(s) (ii) AMS subject classification.

The Bulletin is published Bi-monthly, in February, April, June, August, October, December.
Subscription rate per volume for 1994 Rs. 325/- (INDIA) or $ 125.00 (FOREIGN)
Subscription rate per back volume Rs. 550/- or $ 125.00 (FOREIGN)
The News Bulletin is published monthly.
Subscription rate per volume for 1994 Rs. 36/- (INDIA) or $ 36.00 (FOREIGN)

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Declaration No. 111/88

Digitized by Arya Samaj Foundation Chennai and eGangotri

BULLETIN OF THE CALCUTTA MATHEMATICAL SOCIETY

Volume 86 Number 6 December 1994

CONTENTS

130271

M. C. MUKHERJEE AND A. K. CHONGDAR
A Class of Bilateral Generating Functions of Modified Bessel Polynomials 477

H. M. EL-SHERBINY AND M. A. ABDEL-RAZEK
On the Well-Posedness of a Dissipative RLW Class of Equations 481

N. H. ABDEL-ALL
Quasi-Hyperbolic Manifolds 491

B. M. PANDEYA
Small M-Projective Modules and Submodules of Pseudo Projective Module 497

B. C. DHAGE
On Continuity of Mappings in D-Metric Spaces 503

M. AL-BAALI
A Survey of Classes of Self-Scaling Variable Metric Algorithms 509

U. C. DE AND J. C. GHOSH
On Pseudo W_2-Symmetric Manifolds 521

IKUKO MIYAMOTO
Inclusion Theorems for Absolute Nörlund Summability 527

A. K. SINGH AND U. S. NEGI
On Projective Motion in a Kaehlerian Space with Recurrent Curvature 531

S. DEY AND P. MAHATO
Effective Elastic Coefficients in an Initially Stressed Medium and Propagation or Rayleigh Waves in such a Half Space Bounded by a Rigid Layer 537

E. M. E. ZAYED AND S. F. M. IBRAHIM
Eigenfunction Expansions for a Regular Right-Definite Eigenvalue Problem of Order 2N with an Eigenparameter in the Boundary Conditions 547

RUPA BANDYOPADHYAY AND A. P. BAISNAB
Semi-Closure in a Bitopological Space 555

YU BINGJUN
The Natural Partial Order on a Semigroup 565

LIBRARY
Gurukul Kangri Vishwavidyalaya
HARDWAR

Per volume : (For 1994) Rs. 325.00 (India), or $ 125.00 (Foreign)
Subscription Rate per back volume : Rs. 550.00 (India) or $ 125.00
PRINTED IN INDIA
PRINTED AND PUBLISHED BY DR. R. N. SEN ON BEHALF OF
CALCUTTA MATHEMATICAL SOCIETY
AE 374, SECTOR I, SALT LAKE CITY, CALCUTTA 700 064, INDIA AND PRINTED AT
BOOKS & ALLIED (P) LTD., 8/1 CHINTAMONI DAS LANE, CALCUTTA 700 009
EDITORIAL SECRETARY : DR. C. G. CHAKRABARTI.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

BULLETIN of
Calcutta Mathematical Society

Bi-monthly
internationally reputed
Journal of Mathematics and
Mathematical Sciences.
85 years of continuous publications.
Circulated all over the world.

NEWS BULLETIN of
Calcutta Mathematical Society

A sister Journal of Bulletin of Calcutta Mathematical Society publishes semi-popular (without complicated mathematics) review articles, articles of pedagogic interest, book reviews, Educational news and academic activities of Calcutta Mathematical Society.

Communications should be made with:

Dr. C. G. Chakrabarti
Editorial Secretary.
AE 374, Sector I, Salt Lake City
Calcutta 700 064 (INDIA)

Review Bulletin of The Calcutta Mathematical Society

Bi-annual Review Journal in addition to its Bulletin and News Bulletin.
The good review articles, the lectures delivered in the Society
which are not out and out research articles nor popular articles
are published in this Journal and circulated internationally.

The Review Bulletin is published Bi-annually, in June and December.

Suscription rate per volume for 1993 Rs. 200.00 (INDIA) or $ 75.00 (FOREIGN).

In sending manuscript of papers, please supply the following information
(i) Major field(s) (ii) AMS subject classification.

The Bulletin is published Bi-monthly, in February, April, June, August, October, December.

Subscription rate per volume for 1994 Rs. 325.00 (INDIA) or $125.00 (FOREIGN)

Subscription rate per back volume Rs. 550.00 or $125.00 (FOREIGN)

The News Bulletin is published monthly.

Subscription rate per volume for 1994 Rs. 36.00 (INDIA) or $36.00 (FOREIGN)

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc., 86, 193–200 (1994)

ON THE GEOMETRY OF SPECIFIC TYPES OF POINTS

M. BELTAGY

(*Received 2 November 1992*)

Abstract. In this paper local farthest as well as local exposed points of subsets in Riemannian manifolds without focal points are defined. Relation between convexity and these types of points is studied. Applications in both Euclidean and hyperbolic spaces are given.

1. Introduction. In (Kelly and Weiss, 1979, Valentine 1964), the concept of farthest and exposed points of a subset in a general linear metric space is given. The concept of farthest points of subsets of Euclidean and hyperbolic spaces is considered as a sufficient condition for convexity (Beltagy 1991). The main result in (Beltagy 1991) has been generalized finally to include the study of the same concept for subsets in a complete simply connected C^∞ Riemannian manifold $\widetilde{W}$ without focal points. Actually, the following result has been proved in (Beltagy 1993).

THEOREM (1.1). *Let $M \subset \widetilde{W}$ be a smooth compact hypersurface without boundary. Assume that each point $p \in M$ is a farthest point with respect to some point $x = x(p) \in \widetilde{W}$. Then M is the boundary of a strictly convex body of $\widetilde{W}$.*

An example is given in (Beltagy 1993) to show that the converse of the above theorem (1.1) is not generally true.

In the present work we try to replace the word "farthest" in theorem (1.1) by "local farthest". Moreover, we try to study the relation between convexity and local exposed points as this study is very close to that of local farthest points.

From now on let W (resp. $\widetilde{W}$) denote a complete simply connected C^∞ Riemannian manifold without conjugate (resp. focal) points. M is a general smooth manifold and $K_M(x)$ is the sectional curvature of M at the point $x \in M$. E^n and H^n will denote the Euclidean and hyperbolic n-spaces, respectively. S^n will denote the unit n-sphere. $\bar{B}, B^\circ$ and ∂B will denote the closure, the interior and the boundary of the subset B, respectively. For the couple of points $p, q \in W$, let $d(p,q)$ denote the distance from p to q while $d(p,A)$ is the Hausdorff distance from the point p to the subset $A \subset W$. For the pair of points $p, q \in W, [pq]$ (resp. (pq)) will denote the geodesic segment joining p and q with end points included (resp. deleted). $B(p,\lambda)$ will denote the geodesic ball of centre p and radius $\lambda > 0$ where $S(p,\lambda) = \partial B(p,\lambda)$ is the bounding geodesic sphere. Finally, all manifolds and maps are sufficiently smooth for discussions to make sense. All curves are parametrized by arc-length.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

For the main properties of W and $\widetilde{W}$ we refer the reader to (Bolton 1982), (Eschenburg 1977), (Goto 1978), (Heintze and Hof 1977) and (Kobayashi and Nomizu 1969).

For the element v of the unit sphere bundle SW of W let us write $D_v = \{x \in W : b_v(x) > 0\}$, where $b_v : W \to \mathbb{R}$ is the Busemann function of v, to denote the open horodise characterised by v. $H_v = \partial D_v = \{x \in W : b_v(x) = 0\}$ is the bounding horosphere (Eschenburg 1977).

The following remarkable results are essential for the present work.

THEOREM (1.2). *Let $f : M \to E^{n+1}$ (resp. M^{n+1}) be a smooth isometric immersion of the smooth closed connected Riemannian n-manifold $M(n \geqq 2)$ with sectional curvature $K \geqq 0$ (resp. $K \geqq -1$), then f is an imbedding and $f(M)$ is the boundary of a convex body. (By the word "closed" we mean compact without boundary).*

For $f(M) \subset E^{n+1}$ (Sacksteder 1960) and for $f(M) \subset H^{n+1}$ (Beltagy 1982).

2. Main Results. In a similar way to that of (Kelly and Weiss 1979) we can define both farthest (resp. exposed) and local farthest (resp. local exposed) points of subsets of W as follows.

DEFINITION (2.1). (a) *A point $p \in W$ has a farthest point m in a subset $S \subset W$ if* (i) $m \in S$ (ii) $d(p,m) \geqq d(p,x)$ *subset* $\forall\, x \in S$.

(b) *A point $p \in W$ has a local farthest point m in a subset $S \subset W$ if there exists a neighborhood U about m such that m is a farthest point of p in the subset $\overline{S \cap U}$.*

DEFINITION (2.2). (a) *A point $p \in S \subset W$ is a v-exposed point of S if there exists an element $v \in SW$ such that* (i) $S \subset \bar{D}_v$; (ii) $S \cap H_v = \{p\}$.

(b) *A point $p \in S \subset W$ is a v-local exposed point of S if there exists a neighborhood U about p such that p is a v-exposed point of $U \cap S$.*

Clearly, each farthest (resp. v-exposed) point of a subset $S \subset W$ is a local farthest (resp. v-local exposed) but the converse is not generally true. Also, for a subset $S \subset W$ if $p \in S$ is a v-exposed point of S then p cannot be a $(-v)$-exposed point.

The following facts concerning horospheres of $\widetilde{W}$, which we use intensively throughout this work, are sufficient motivations to prefer horospheres to any other hypersurfaces of $\widetilde{W}$ (such as geodesic hypersurfaces) to be used as characteristic hypersurfaces (Bolton 1982).

(i) Horospheres determined by a certain Busemann function form a parallel family of hypersurfaces whose orthogonal trajectories are geodesics.

(ii) For an element $v \in SW, H_v$ is the limit of a sequence of geodesic spheres, i.e., H_v is itself a geodesic sphere of infinite radius. This fact enables us to relate exposed points with farthest points as it will be shown next.

(iii) In general $\widetilde{W}$, totally geodesic hypersurfaces do not exist.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(iv) A geodesic hypersurface in $\widetilde{W}$ at a point is not necessarily geodesic at any other point in contrary to horospheres.

It is an easy exercise to show that if the point $p \in W$ has the point $x \in M$ of the smooth submanifold $M \subset W$ as a farthest (local farthest) point then the geodesic through p and x meets M at x orthogonally. In addition for a subset $A \subset W$ no interior point of A could be either farthest or local farthest point. Consequently, all farthest (local farthest) points of $A \subset W$ are boundary points. Same thing is valid for exposed (local exposed) points. Moreover, every farthest (resp. v-exposed) point of $A \subset W$ is local farthest (resp. v-local exposed) but the converse is not generally true.

Situation is completely different for subsets in manifolds with conjugate points. For instance the point $-a \in S^n \subset E^{n+1}$ has the point $a \in B(a, \lambda) \subset S^n, \lambda < \pi$, as a farthest point of $\overline{B(a, \lambda)}$ while a is not a boundary point of $\overline{B(a, \lambda)}$.

In the following we give a brief word on convexity as the main purpose of the present work is to relate convexity with the above-mentioned concepts of farthest and exposed points.

A subset B of the Riemannian manifold M is convex if for each pair of points $p, q \in B$ there exists a unique minimal geodesic segment $[pq]$ such that $[pq] \subset B$. When dealing with a subset $B \subset W$, the word "a unique minimal geodesic segment" should be replaced by "the geodesic segment". A body $B \subset M$ is a compact connected set in M which is the closure of an open set. The boundary of an open convex set in M is a convex hypersurface. A convex subset $B \subset M$ is called strictly convex if ∂B contains no geodesic segments. The boundary of a strictly convex body $B \subset M$ is called convexoid. If in addition $K_{\partial B} > K_M$ then ∂B is called ovaloid.

For an open convex subset $B \subset W$ we can show that $\bar{B}$ is also convex while the same thing is not valid in general Riemannian manifolds. For example, the open geodesic ball $B(p, \frac{\pi}{2}) \subset S^n$ is convex but $\overline{B(p, \frac{\pi}{2})}$ is non-convex. Other types of convexity in general Riemannian manifolds such as strong convexity and weak convexity could be found in (Alexander 1978).

It is an easy exercise to show that the intersection of two (or more) convex subsets in M is a convex subset. Same property is no longer true in the case of union of two subsets.

The following basic results are essential for the present work. Proofs could be found in (Beltagy 1990, 1991).

LEMMA (2.3). *In* $\widetilde{W}$

(i) *closed geodesic balls are strictly convex bodies. (i.e., geodesic spheres of finite radii are convexoids)*

(ii) *closed horodiscs are convex subsets. (i.e., horospheres are convex hypersurfaces).*

LEMMA (2.4). *For a closed convex subset $B \subset W$ with smooth boundary hypersurface ∂B each tangent geodesic γ to ∂B has the property $\gamma \cap B^\circ = \phi$.*

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In the light of lemma (2.4) we have the following:

COROLLARY (2.5). *Let $p \in \widetilde{W}$ be an arbitrary point and $v \in \widetilde{W}_p$ an arbitrary unit vector. Let $\gamma : (-\infty, \infty) \to \widetilde{W}$ be a maximal geodesic such that $\gamma(0) = p$ and $\gamma'(0)$ is perpendicular to v. Then $\gamma \subset \widetilde{W}/(D_v \cup D_{-v})$.*

The following remarkable result concerning convexity has been proved by H. Karcher in 1968.

PROPOSITION (2.6). A connected open subset B of a complete Riemannian manifold M is convex if and onlv if B possesses a local support element at every boundary point and does not intersect its own cut locus.

For the definition of the local support element of $B \subset M$ (Alexander 1978 and Karcher 1968).

If we replace M by W in the last proposition (2.6) the word "cut locus" should be disregarded completely and consequently we have the following form:

COROLLARY (2.7). *A connected open subset $B \subset W$ is convex if and only if B possesses a local support element at every boundary point.*

In the light of lemma (2.4) we might replace the word "local" in the above corollary (2.7) by the word global.

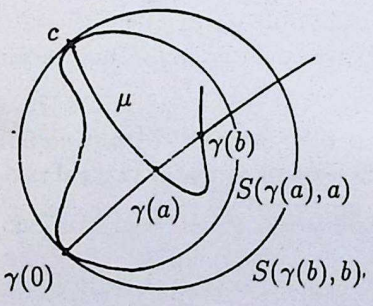
LEMMA (2.8). *Let $\gamma : [0, \infty) \to W$ be a geodesic ray and $a, b \in \mathbb{R}$ such that $b > a > 0$. Then*

$$\text{(i)}\ \overline{B(\gamma(b), b)} \supset \overline{B(\gamma(a), a)}; \quad \text{(ii)}\ S(\gamma(a), a) \cap S(\gamma(b), b) = \{\gamma(0)\}.$$

Proof. The first part is direct as if we consider an arbitrary point $x \in \overline{B(\gamma(a), a)}$ and use the triangular property of the distance function d we have

$$d(x, \gamma(b)) \leqq d(x, \gamma(a)) + d(\gamma(a), \gamma(b)) \leqq a + (b - a) = b$$

which means that $x \in \overline{B(\gamma(b), b)}$.



In contrary to (ii) assume that $S(\gamma(b), b) \cap S(\gamma(a), a) = \{\gamma(0), c\}$. Consider the geodesic ray $\mu : [0, \infty) \to W$ such that $\mu(0) = c$ and $\mu(a) = \gamma(a)$. By uniqueness of geodesics we have that $\mu(b) = \gamma(b)$ which means that there are two different geodesic segments $\gamma|_{[a,b]}$ and $\mu|_{[a,b]}$ from $\gamma(a)$ to $\gamma(b)$ (See Fig. (1)). This contradicts the fact that each point of W is a pole (Kobayashi and Nomizu 1969).

Letting $b \to \infty$ in the above lemma (2.3) we obtained the following result.

COROLLARY (2.9). *Let $\gamma : [0, \infty) \to W$ be a geodesic ray such that $\gamma'(0) = v$. Then for each $a > 0$.*

$$\text{(i)}\ B(\gamma(a), a) \subset D_v; \quad \text{(ii)}\ H_v \cap S(\gamma(a), a) = \{\gamma(a)\}.$$

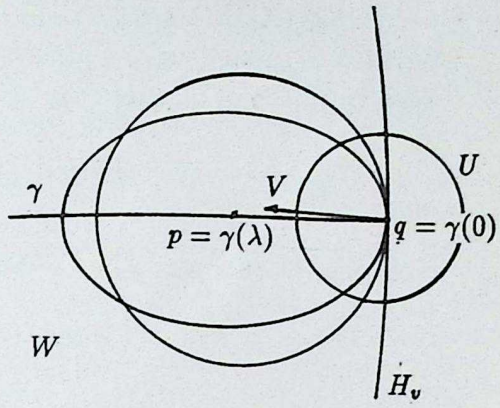
CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

According to the above corollary (2.9) we conclude that each point of $S(a,\lambda) \subset W$ is an exposed point of $S(a,\lambda)$ (or $\overline{B(a,\lambda)}$) for arbitrary finite $\lambda > 0$.

PROPOSITION (2.10). *For a subset $A \subset W$ every local farthest point is local exposed.*

Proof. Assume that $q \in A$ is a local farthest point of A with respect to some point $p \in W$. Then there exists a neighbourhood U about q such that $\overline{U \cap A} \subset \overline{B(p,\lambda)}$ where $\lambda = d(p,q)$. Let $\gamma : [0,\infty) \to W$ be a geodesic ray such that $\gamma(0) = q$, $\gamma(\lambda) = p$ and $\gamma'(0) = v$. By corollary (2.9) we have that q is a v-exposed point of $\overline{U \cap A}$ and consequently q is a local exposed point of A (See Fig. (2)).



The converse of the above proposition (2.10) is not necessarily true as indicated in the following example.

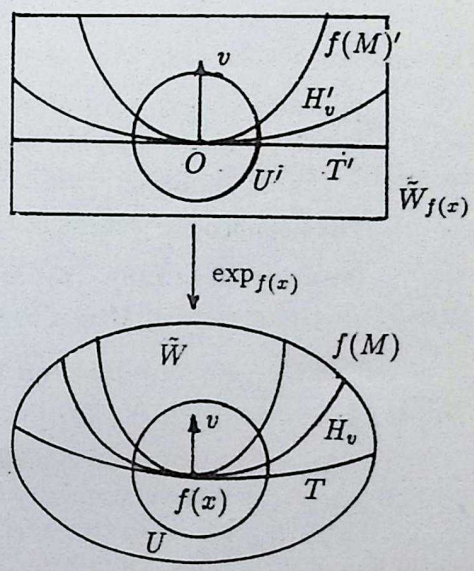
EXAMPLE (2.11). Let $A \subset E^2$ be the upper planar area bounded by the curve $y = x^4$. The origin 0 is a $(0,1)$-local exposed point of A while the same point 0 is not a local farthest point of A with respect to any point of E^2 of the form $(0,a)$. The reason is that if we assume that 0 is a local farthest point of A with respect to some $(0,a) \in E^2$, then the curvature K of the curve $y = x^4$ will satisfy $K \geqq \frac{1}{a} > 0$ at 0 which is a contradiction.

PROPOSITION (2.12). *Let $f : M \to \widetilde{W}$ be a smooth immersion of codimension 1 such that $f(x), x \in M$, is a v-local exposed point of $f(M)$. Then*

$$K_{f(M)}(f(x)) \geqq K_{H_v}(f(x)) \geqq K_{\widetilde{W}}(f(x)).$$

Proof. Let T be the geodesic hypersurface of $\widetilde{W}$ at $f(x)$ tangent to H_v at the same point. By lemma (2.3) and under the assumptions of the proposition we have Fig. (3) for a sufficiently small neighborhood U about $f(x)$. Since the second fundamental form of M at x equals the Hessian of the height function $h_{f(M)}$ of $f(M)$ at $f(x)$ with respect to the hyperplane $T' = \exp^{-1}_{f(x)} T$ (Kobayashi and Nomizu 1979) we have directly that for every $X \in \exp^{-1}_{f(x)} T$

$$X^2 h_{f(M)} \geqq X^2 h_{H_v} \geqq 0. \qquad (2.1)$$



Taking $\{X_1, X_2, \ldots, X_n\} \subset \exp^{-1}_{f(x)} T$ to be the set of orthonormal eigenvectors of the second fundamental form of $f(M)$ at $f(x)$ and using Gauss equation we have for $1 \leqq i, j \leqq n$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$K_{f(M)}(X_i, X_j) = K_{\widetilde{W}}(X_i, X_j) + \left(X_i^2 h_{f(M)}\right)\left(X_j^2 h_{f(M)}\right) \tag{2.2}$$
$$K_{H_v}(X_i, X_j) = K_{\widetilde{W}}(X_i, X_j) + \left(X_i^2 h_{H_v}\right)\left(X_j^2 h_{H_v}\right) - (X_i X_j H_v). \tag{2.3}$$

From (2.1)– (2.3) we have that

$$K_{f(M)}(f(x)) \geqq K_{H_v}(f(x)) \tag{2.4}$$

Applying the same procedure to both H_v and T at $f(x)$ we have that

$$K_{H_v}(f(x)) \geqq K_T(f(x)). \tag{2.5}$$

By inequalities (2.4) and (2.5) and since $K_{\widetilde{W}} = K_T$ at $f(x)$ we obtain the require result.

In the same way of proof we have the following result.

PROPOSITION (2.13). *Let $f : M \to \widetilde{W}$ be a smooth immersion of codimension 1 suc that $f(x)$, for $x \in M$, is a local farthest point of $f(M)$ with respect to some $p \in \widetilde{W}$. Then*

$$K_{f(M)}(f(x)) \geqq K_{S(p,\lambda)}(f(x)) \geqq K_{H_v}(f(x)) \geqq K_{\widetilde{W}}(f(x)),$$

where $\lambda = d(p, x)$.

Since W is diffeomorphic to E^n, we can write Jordan-Brouwer separation theorem i the following general form (Alexander 1978).

THEOREM (2.14). *Let M be an imbedded closed connected hypersurface in W. The $W \backslash M$ consists of two components each with boundary M. Moreover, one of the components bounded.*

The bounded component in the above theorem will be denoted by Int (M).

PROPOSITION (2.15). *Let M be an imbedded closed connected smooth hypersurface $\widetilde{W}$ such that each point of M is a local exposed point of $\overline{Int(M)}$.* Then M is a convexoid.

Proof. We show that $\overline{Int(M)}$ is a strictly convex subset of $\widetilde{W}$. Firstly, we show th $\overline{Int(M)}$ is a convex body of $\widetilde{W}$.

Since each point of M is a local exposed point of $\overline{Int(M)}$, then for an arbitrary poi $p \in M$ there exists a sufficiently small neighborhood U about p such that $\overline{Int(M) \cap U}$ supported by H_v, where $v \in \widetilde{W}_p$ is the unit vector normal to M at p in the interior directio Using Fig. (3) we conclude that the open subset $Int(M)$ has a local support element at ea

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

point $p \in M$. Applying Karcher's result (proposition (2.6)) we have directly that $Int(M)$ (or $\overline{Int(M)}$) is a convex subset of $\widetilde{W}$.

Finally, assume that $\overline{Int(M)}$ is convex but not strictly convex. Consequently, there exists a geodesic segment $[ab] \subset M$ for $a, b \in M$. Let p be the middle point of $[ab]$. Since p is a local exposed point of $\overline{Int(M)}$, then for a sufficiently small neighborhood U about p the geodesic segment $\gamma = \overline{[ab]} \cap U$ which is tangent to H_v at p has the property that $\gamma \cap D_v^o \neq \phi$ for v as mentioned above. This is now a contradiction with lemma (2.4).

Since each local farthest point of a subset of $\widetilde{W}$ with respect to some other point of $\widetilde{W}$ is a local exposed point (proposition (2.10)), then we can replace the word "local exposed" in the above proposition (2.15) by the word "local farthest" and the result will be still valid.

Taking $\widetilde{M}(c), c = 0, -1$, instead of $\widetilde{W}$ in proposition (2.15) we have the following results.

PROPOSITION (2.16). *Let $f : M \to \widetilde{M}(c)$ be an immersion of the smooth closed connected n-manifold $M(n \geqq 2)$ into the $(n+1)$-non-elliptic space form $\widetilde{M}(c)$ such that each point of $f(M)$ is a local exposed point of $f(M)$. Then f is an imbedding and $f(M)$ is a convexoid.*

Proof. By proposition (2.12) we have directly that at each point $x \in M$, $K_{f(M)}(f(x)) \geqq c$. The proof now becomes a direct conclusion in the light of proposition (2.15) and theorem (1.2).

Taking into account that each local farthest point $x \in M \subset \widetilde{M}(c)$, $c = 0, -1$, with respect to $p \in \widetilde{M}(c)$ is a local exposed point of M (proposition (2.10)) and by proposition (2.13) $K_M(x) \geqq K_{S(p,\lambda)}(x) > 0$, $\lambda = d(p, x)$ (Beltagy 1982), (Cecil and Ryan 1979), then the above proposition (2.16) leads directly to the following result.

PROPOSITION (2.17). *Let $f : M \to \widetilde{M}(c)$ be an immersion of the smooth closed connected n-manifold $M(n \geqq 2)$ into the $(n+1)$-non-elliptic space form $\widetilde{M}(c)$ such that each point of M is a local farthest point of M with respect to some point of $\widetilde{M}(c)$. Then f is an imbedding and $f(M)$ is an ovaloid.*

The proof of this proposition depends basically on theorem (1.2).

Notice that in a space form $\widetilde{M}(c)$ and for an arbitrary finite $\lambda > 0$ (Beltagy, 1982), (Cecil and Ryan 1979)

$$K_{S(p,\lambda)} = \begin{cases} \lambda^{-2} & , \quad c = 0 \\ \operatorname{cosec}^2 \lambda & , \quad c = 1 \\ \operatorname{cosech}^2 \lambda & , \quad c = -1. \end{cases}$$

REMARK. When dealing with global exposed and global farthest points the word "connected" should be neglected. (Beltagy 1993).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

References

Alexander, S. (1978) : Local and global convexity in complete Riemannian manifolds, *Pacific J. Math., 70,* 283.

Beltagy, M. (1982) : Immersions into manifolds without conjugate points, *Ph.D. Thesis,* Durham Univ., U.K.

Beltagy, M. (1990) : Foot points and convexity in manifolds without conjugate points, *Bull. Cal. Math. Soc., 82,* 338.

Beltagy, M. (1991) : On the geometry of foot and farthest points, *Jour. Inst. Math. & Comp. Sci.,* 2, 159.

Beltagy, M. (1993) : On convexity and farthest points, *Bull. Cal. Math. Soc., 85,* 297.

Bolton, J. (1982) : Tight immersions into manifolds without conjugate points. *Quart. J. Math. Oxford, 23,* 159.

Cecil, T. E. and **Ryan, P. J.** (1979) : Distance functions and umbilic submanifolds of hyperbolic space, *Nagoya Math. J., 74,* 67.

DoCarmo, M. P. and **Lima, E.** (1969) : Isometric immersions with semi-definite second quadratic forms, *Arch. Math. (Basel) 20,* 173.

Eschenburg, J. H. (1977) : Horospheres and the stable part of the geodesic flow, *Math. Z., 153,* 237.

Goto, M. S. (1978) : Manifolds without focal points. *J. Diff. Geom., 13,* 341.

Heintze, E. and **Hof, H. C. I.** (1977) : Geometry of horospheres, *J. Diff. Geom., 12,* 481.

Karcher, H. (1968) : Schnittort and konvexe Mangen in vollstundigen Riemannschen Mannigfaltigkeiten, *Math. Ann., 117,* 105.

Kelly, P. J. and **Weiss, M. L.** (1979) : *Geometry and convexity,* John Wiley & Sons, Inc., New York.

Kobaysashi, S. and **Nomizu, K.** (1969) : *Foundations of differential geometry, Interscience Publish,* II.

Sacksteder, R. (1960) : On hypersurfaces with no negative sectional curvatures, *Amer, J. Math., 82,* 609.

Valentine, F. A. (1964) : *Convex sets,* McGraw-Hill, New York.

MATHEMATICS DEPARTMENT
FACULTY OF SCIENCE
TANTA UNIVERSITY
TANTA
EGYPT

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc., 86, 201–204 (1994)

INFINITE SERIES INVOLVING BESSEL FUNCTIONS OF THE SAME ORDERS

SHAHWAR F. RAGAB

(*Received 16 September 1991*)

Abstract. In this paper infinite series expansions of Bessel Functions in terms of the other Bessel functions of the same order are obtained. This work can be extended to any other different orders of Bessel functions as my definitions for these functions.

1. Introduction. Bessel functions are used in many applications such as mechanical electric and physical applications. In chapter 2 formula required in the proofs are written. In chapter 3 the main two theorems for these expansions are deduced and then we can conclude the expansion which are the aim of this paper. I mention for $\underset{\alpha,\beta}{J}(x)$ which is known as P. Humbert function with the expressions Bessel function of second order, while $\underset{\alpha,\beta,\gamma}{J}(x)$ is Bessel function of third order and so on.

Formulae required in the proofs of part II

$$\frac{1}{2\Pi i}\int \frac{\Gamma^{(\xi)}\prod_{r=1}^{p}\Gamma^{(\alpha_1-\zeta)}}{\prod_{s=1}^{q}\Gamma^{(P_s-\xi)}} Z^{\zeta}d^{\zeta} = E\left(\begin{matrix} p; & \alpha_r : z \\ q; & P_s \end{matrix}\right), \tag{1}$$

where E function on the R.H.S. of (1) is the Mac Robert's E-function where definitions and Properties are to be found in (Ragab 1989, pp. 348–382) and in (Mac Robert 1962, pp., 203–206), the integral in (1) is known as Barnes integral.

If in $p \leq q+1$ in (1) we get (2).

$$E(p;\alpha_r;q;P_s:z) = \frac{\Gamma^{(\alpha_1)}\ldots\Gamma^{(\alpha_p)}}{\Gamma^{(P_1)}\ldots\Gamma^{(P_s)}} F\left(\begin{matrix} \alpha_1\ldots, & \alpha_p; -\frac{1}{2} \\ P_1,\ldots, & P_q \end{matrix}\right) \tag{2}$$

$${}_0F_1(;\gamma+1;z) = \gamma! z^{\frac{\gamma}{2}} J_\gamma\left(\frac{2}{\sqrt{z}}\right) \tag{2'}$$

$$\underset{\mu,\gamma}{J}(z) = \frac{z^{\mu+\gamma}}{3^{\mu+\gamma}\Gamma^{(\mu+1)}\Gamma^{(\gamma+1)}}\,{}_0F_2\left(\begin{matrix} \\ ;\mu+1,\gamma+1 \end{matrix}; \frac{-z^3}{27}\right) \tag{3}$$

$$\underset{\mu,\gamma,\lambda}{J}(z) = \frac{z^{\mu+\gamma+\lambda}}{4^{\mu+\gamma+\lambda}\Gamma^{(\mu+1)}\Gamma^{(\gamma+1)}\Gamma^{(\lambda+1)}}\,{}_0F_3\left(\begin{matrix} \\ ;\mu+1,\gamma+1,\lambda+1 \end{matrix}; \frac{-z^4}{256}\right) \tag{4}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\underset{\mu,\gamma,\lambda,R}{J}(z) = \frac{z^{\mu+\gamma+\lambda+R}}{5^{\mu+\gamma+\lambda+R}\Gamma^{(\mu+1)}\Gamma^{(\gamma+1)}\Gamma^{(\lambda+1)}\Gamma^{(R+1)}}\,{}_0F_4\left(\begin{matrix} \\ ;\mu+1,\gamma+1,\lambda+1,R+1 \end{matrix}\;;-\left(\tfrac{z}{5}\right)^5\right) \tag{5}$$

(4), (5) are my definites and it can be extended.

$$F\left(\begin{matrix}\alpha & \beta; & 1\\ & \gamma & \end{matrix}\right) = \frac{\Gamma^{(\gamma)}\Gamma^{(\gamma-\alpha-\beta)}}{\Gamma^{(\gamma-\alpha)}\Gamma^{(\gamma-\beta)}} \tag{6}$$

which is Gauss's theorem, provided that

$$R(\gamma-\alpha-\beta) > 0.$$

Evaluation of the series

The first theorem to be proved is

$$\sum_{n=0}^{\infty}\frac{(a;n)(b;n)}{n!}E\left(\begin{matrix} c-a, c-b, \alpha_1,\ldots,\alpha_p; z \\ c+n, c-a-b, \mathcal{S}_1,\ldots,\mathcal{S}_q \end{matrix}\right) = E(p;\alpha_r : q;\rho_s : z) \tag{7}$$

$$R(c-a-b) > 0$$

to prove (7) substitute from (1) on the left hand side and get

$$\frac{1}{2\Pi i}\sum_{n=0}^{\infty}\frac{(a;n)(b;n)}{n!}\int\frac{\Gamma^{(\xi)}\Gamma^{(c-a-\xi)}\Gamma^{(c-b-\xi)}\prod_{r=1}^{p}\Gamma^{(\alpha_r-\xi)}}{\Gamma^{(c+n-\xi)}\Gamma^{(c-a-b-\xi)}\prod_{s=1}^{q}\Gamma^{(\mathcal{S}_s-\xi)}}z^{\xi}\,d\xi.$$

Here change the order of summation and integration, so getting

$$\frac{1}{2\Pi i}\int\frac{\Gamma^{(\xi)}\Gamma^{(c-a-\xi)}\Gamma^{(c-b-\xi)}\prod_{r=1}^{p}\Gamma^{(\alpha_r-\xi)}z^{\xi}}{\Gamma^{(c-a-b-\xi)}\Gamma^{(c-\xi)}\prod_{s=1}^{q}\Gamma^{(\mathcal{S}_s-\xi)}}$$

$$X\,{}_2F_1\left(\begin{matrix} ab; & 1\\ c-\xi & \end{matrix}\right).$$

Now use (6) to sum the ${}_2F_1$ and the result follows.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In (7) take $p = 0$ and use (2) with $q = 2$

$$\mathcal{S}_1 = c - a, \quad \mathcal{S}_2 = c - b$$

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n! \Gamma^{(c-a-b)} \Gamma^{(c)}} {}_0F_2 \left(\begin{matrix} \\ ; c+n, c-a-b \end{matrix} ; -\frac{1}{z} \right)$$

$$= \frac{1}{\Gamma^{(c-a)} \Gamma^{(c-b)}} {}_0F_2 \left(\begin{matrix} \\ c-a, c-b \end{matrix} ; -\frac{1}{z} \right).$$

Now put $z = \frac{27}{x^3}$, use (3) and get

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!} \left(\frac{3}{x} \right)^n \underset{c+n-1, c-a-b-1}{J(x)} = \underset{c-a-1, c-b-1}{J(x)} \tag{8}$$

(8) is infinite series expansions between two P' Humbert functions or Bessel function of order two.

In (7) $P = 0, q = 3$, with $\mathcal{S}_1 = c - a, \mathcal{S}_2 = c - b, \mathcal{S}_3 = a + b$ with $z = \frac{256}{x^4}$, using (4) and get

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!} \left(\frac{4}{x} \right)^n \underset{c+n-1, c-a-b-1, a+b-1}{J(x)} = \underset{c-a-1, c-b-1, a+b-1}{J(x)} \tag{9}$$

which is series expansion involving Bessel function of third order.

In (7) take $p = 0, q = 4$, with $\mathcal{S}_1 = c - a, \mathcal{S}_z = c - b, \mathcal{S}_3 = a + b, \mathcal{S}_4 = a - b$ with $z = \left(\frac{5}{x}\right)^5$ we obtain an infinite series expansion involving Bessel function of order four from (5).

Then we can extend these formulae for any order of Bessel function in the same way.

The second main theorem to be proved is

$$\sum_{n=0}^{\infty} \frac{x^{-n}}{n!} E \left(\begin{matrix} p; & \alpha_r + n : & z \\ q; & \mathcal{S}_s + n & \end{matrix} \right) = E \left(\begin{matrix} p; & \alpha_r : & \frac{xz}{x-z} \\ q; & \mathcal{S}_s & \end{matrix} \right) \tag{10}$$

if z and x are real and positive and $\left|\frac{z}{x}\right| < |$, to prove (10) use the Barnes integral in (1) and get

$$\text{L.H.S. of (10)} = \frac{1}{2\Pi i} \int \Gamma^{(\xi)} \frac{\prod_{r=1}^{p} \Gamma^{(\alpha_r - \xi)}}{\prod_{s=1}^{q} \Gamma^{(\mathcal{S}_s - \xi)}} z^{\xi}_{0} F_1 \left(\xi; \frac{z}{x} \right) d\xi = \frac{1}{2\Pi i} \int \Gamma^{(\xi)} \frac{\prod_{r=1}^{p} \Gamma^{(\alpha_r - \xi)}}{\prod_{s=1}^{q} \Gamma^{(\mathcal{S}_s - \xi)}} \left(\frac{1}{z} - \frac{1}{x} \right)^{-\xi} d\xi$$

$$= E \left(\begin{matrix} \alpha_1, \ldots, \alpha_p; & \frac{x\ z}{x-z} \\ \mathcal{S}_1, \ldots, \mathcal{S}_q & \end{matrix} \right) = \text{R.H.S. of (10)}$$

thus (10) is proved.

In (10) write $\frac{xz}{x+y}$ for z and get

$$\sum_{n-0}^{\infty} \frac{x^{-n}}{n!} E \left(\begin{matrix} p; & \alpha_r + n : & \frac{xy}{x+y} \\ q; & \mathcal{S}_s + n & \end{matrix} \right) = E \left(\begin{matrix} p; & \alpha_r : & y \\ q; & \mathcal{S}_s & \end{matrix} \right) \tag{11}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In (11) replace x and y by $\frac{4}{x^2}, \frac{4}{y^2}$, use (2') with $p=0, q=1, \mathcal{S}_1=\gamma+1$, we get

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{x^2}{2\sqrt{x^2+y^2}}\right)^n \underset{\gamma+n}{J}(\sqrt{x^2+y^2}) = \left(\frac{\sqrt{x^2+y^2}}{y}\right)^{\gamma} \underset{\gamma}{J}(y) \tag{12}$$

this relates Bessel function of order one which is the known Bessel function.

In (11) replace x and y by $\frac{27}{x^3}, \frac{27}{y^3}$, use (3) with $p=0, q=2, \mathcal{S}_1=\gamma+1, \mathcal{S}_2=\mu+1$, we get

$$\sum_{n=0}^{\infty}\left(\frac{3}{x^3}\right)^{-n}\left(\frac{1}{3\sqrt{x^3+y^3}}\right)^{\mu+\gamma+2n} \underset{\mu+n,\gamma+n}{J}(3\sqrt{x^3+y^3})$$
$$= \left(\frac{1}{y}\right)^{\mu+\gamma} \underset{\mu,\gamma}{J}(y) \tag{13}$$

Again in (11) take $p=0, q=3$, with $\mathcal{S}_1=\gamma+1, \mathcal{S}_2=\mu+1, \mathcal{S}_3=\lambda+1$, replace x and y by $\left(\frac{4}{x}\right)^4, \left(\frac{4}{y}\right)^4$, use (4) we get

$$\sum_{n=0}^{\infty}\left(\frac{4}{x^4}\right)^{-n}\left(\frac{1}{4\sqrt{x^4+y^4}}\right)^{\mu+\gamma+\lambda+3n} \underset{\mu+n,\gamma+n,\lambda+n}{J}(4\sqrt{x^4+y^4})$$
$$= \left(\frac{1}{y}\right)^{\mu+\gamma+\lambda} \underset{\mu,\gamma,\lambda}{J}(y). \tag{14}$$

In the same way we extend the work to any number of infinite series between other Bessel functions of different order.

For example using (5)

$$\sum_{n=0}^{\infty}\left(\frac{5}{x^5}\right)^{-n}\left(\frac{1}{5\sqrt{x^5+y^5}}\right)^{\mu+\gamma+\lambda+\alpha+4n} \underset{\mu,n,\gamma+n,\alpha+n,\lambda+n}{J}(5\sqrt{x^5+y^5}) = \left(\frac{1}{y}\right)^{\alpha+\mu+\gamma+\lambda} \underset{\mu,\gamma,\lambda,\alpha}{J}(y). \tag{15}$$

References

Erdely, A., Mcgnus, M., Oberhetinger, F., and Tricomi (1953) : *Higher Trauscendental Function,* 1 Magraw-Hill New York.

Macrobert, T. M. (1962) : *Functions of a complex variable,* London.

Ragab, F. M. (1962) : Infinite Series of E-Function, *Math-Annalen, 148,* 94.

Ragab, S. F. (1989) : Infinite Series of E-function, *Tamkang Journal of Mathematics, 20,* 3.

ASSISTANT PROFESSOR
MATHEMATICS DEPARTMENT
FACULTY OF ENGINEERING
CAIRO UNIVERSITY
EGYPT
A.R.E.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc., 86, 205–212 (1994)

THE STRESSES IN A ROTATION OF A NON-HOMOGENEOUS COMPOSITE CYLINDER OF ISOTROPIC MATERIAL

A.M.EL-NAGGAR, A.M. ABD-ALLA AND S. M. AHMED

(*Received 10 February 1993*)

Abstract. In this paper, we investigate the effect of non-homogeneity of the material and density on the stresses in a rotation of a non-homogeneous isotropic cylinder containing (i) a homogeneous core and (ii) a rigid core. Those cases have been illustrated and discussed using tables and figures.

1. Introduction. Some problems of plane strain in a non-homogeneous isotropic cylinder were investigated by Chatterjee (1970), Mukhopadhyay (1972) solved a problem on the effects of non-homogeneity on the stresses in a rotation of a non-homogeneous aeolotropic cylindrical shell. On the other hand, Mukhopadhyay (1976) solved the present problem in the case of homogeneous density. Senitskii (1978) investigated a dynamic problem of inhomogeneous anisotropic for cylinder and sphere. This work deals with the latter problems of plane strain in a non-homogeneous isotropic circular cylinder, viz., (i) rotation about its axis of a non-homogeneous circular cylinder containing a homogeneous core, (ii) rotation about its axis of a non-homogeneous circular cylinder containing a rigid core. In both problems, we are assuming that, the elastic properties and density are non-homogeneous. By homogeneous, we shall always mean homogeneous material but non-homogeneous density. In both cases, the stresses and extensions have been calculated in non-homogeneous and homogeneous cases. Finally, a comparison between both cases are clarified by tables and figures.

2. Formulation of the Problem. We use cylindrical co-ordinates (r, θ, z) with the z-axis coinciding with the axis of the cylinder. To consider the strains symmetrical about the z-axis, we have only the radical displacement $u_r = u$ and the longitudinal displacement $u_z = \omega$, which are independent of θ. In plane strain, in the planes perpendicular to the z-axis, u is a function of r alone, and $\omega = \epsilon z$, where ϵ is the constant extension in the direction of the z-axis.

The stresses are given by

$$\begin{aligned}
\tau_{rr} &= \frac{2\mu}{1-2\nu}\left[(1-\nu)\frac{du}{dr} + \nu\left(-\frac{u}{r} + \epsilon\right)\right],\\
\tau_{\theta\theta} &= \frac{2\mu}{1-2\nu}\left[(1-\nu)\frac{u}{r} + \nu\left(\frac{du}{dr} + \epsilon\right)\right],\\
\tau_{zz} &= \frac{2\mu}{1-2\nu}\left[(1-\nu)\epsilon + \nu\left(\frac{du}{dr} + \frac{u}{r}\right)\right],\\
\tau_{\theta z} &= \tau_{zr} = \tau_{r\theta} = 0,
\end{aligned} \tag{1}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

where μ is the rigiditly modulus and ν the position's ratio of the material, and μ is assumed to be a function of r while ν is constant.

Two equations of equilibrium are identically satisfied while the third, concerning the rotation about the z-axis, is given by

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) + \rho\Omega^2 r = 0, \tag{2}$$

where ω is the uniform angular velocity and ρ is the density of the cylinder material.

We characterize the non-homogeneity of the material and density by

$$\mu = \mu_0 e^{-kr^m}, \tag{3}$$

$$\rho = \rho_0 r^{m-2}, \tag{4}$$

where μ_0 and ρ_0 are constants and m is a rational number.

Substituting (1), (3) and (4) in (2) we get

$$\frac{d^2u}{dr^2} + \frac{1}{r}(1 - mkr^m)\frac{du}{dr} - \left(1 + \frac{\nu mk}{1-\nu}r^m\right)\frac{u}{r^2} = \frac{\nu mk\epsilon}{1-\nu}r^{m-1} - \frac{(1-2\nu)\rho_0\Omega^2}{2(1-\nu)\mu_0}r^{m-1}e^{kr^m} \tag{5}$$

Putting

$$u = e^{\frac{x}{2}}\phi(x),$$

where $x = kr^m$ and taking

$$\frac{m}{2} = \frac{\nu}{1-\nu},$$

equation (5) becomes

$$\frac{d^2\phi}{dx^2} + \frac{1}{x}\frac{d\phi}{dx} - \left(\frac{1}{4} + \frac{1}{m^2x^2}\right)\phi = \frac{1}{2} - m^2k\epsilon r^{m-1} + \frac{(m-2)\rho_0\Omega^2}{4\mu_0}r^{m-1}e^{kr^m}. \tag{6}$$

Equation (6) have a closed form solution if $(\frac{1}{m} - \frac{1}{2})$ is an integer (Mukhopadhyay, 1976). We confine our discussion for $m = \frac{2}{3}$ corresponding to $\nu = \frac{1}{4}$, in this case the complete solution of (6) becomes

$$u = A(2r^{-1} - kr^{-1/3})e^{kr^{2/3}} + B(2r^{-1} + kr^{-1/3}) - \frac{\epsilon}{4}r - \frac{3\rho_0\Omega^2}{8\mu_0 k}r\, e^{kr^{2/3}}, \tag{7}$$

where A and B are constants.

From (7), (3), (4) and (1), the stresses are given by

$$\tau_{rr} = \frac{4\mu_0}{r}\left[A\left(kr^{2/3} - \frac{1}{2}k^2r^{4/3} - 1\right) - B\, e^{-kr^{2/3}}\right] - \frac{3\rho_0\Omega^2}{4k}(2 + kr^{2/3}), \tag{8}$$

$$\tau_{\theta\theta} = \frac{4\mu_0}{r}\left[A\left(1 - \frac{1}{3}kr^{2/3} - \frac{1}{6}k^2r^{4/3}\right) + B\left(1 + \frac{2}{3}kr^{2/3}\right)e^{-kr^{2/3}}\right] - \frac{\rho_0\Omega^2}{4k}(6 + kr^{2/3}), \tag{9}$$

$$\tau_{zz} = \frac{2\mu_0 k}{3r^{3/4}}\left[A(1 - kr^{2/3}) + B\, e^{-kr^{2/3}}\right] - \frac{\rho_0\Omega^2}{4k}(3 + kr^{2/3}) + \frac{5}{2}\mu_0\epsilon\, e^{-kr^{2/3}} \tag{10}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The corresponding displacements and stresses in the homogeneous case; homogeneous material and non-homogeneous density of non-homogeneity as given in (4) for $m = \frac{2}{3}$; for $\nu = \frac{1}{4}$ are given by

$$u = \frac{M}{r} + Lr - \frac{3\rho_o\Omega^2}{16\mu_o} r^{5/3}, \tag{11}$$

$$\tau_{rr} = 2\mu_o\left(2L - \frac{M}{r^2}\right) + \mu_o\epsilon_1 - \frac{9}{8}\rho_o\Omega^2 r^{2/3}, \tag{12}$$

$$\tau_{\theta\theta} = 2\mu_o\left(2L + \frac{M}{r^2}\right) + \mu_o\epsilon_1 - \frac{7}{8}\rho_o\Omega^2 r^{2/3}, \tag{13}$$

$$\tau_{zz} = 2\mu_o L + 3\mu_o\epsilon_1 - \frac{1}{2}\rho_o\Omega^2 r^{2/3}, \tag{14}$$

where L and M are constants and ϵ_1 is the constant extension along z-axis; μ_o and ρ_o are the values of μ and ρ in the homogeneous case and ω is the angular velocity about the z-axis. For complete cylinder $M = 0$.

3. Rotation of a circular cylinder containing a homogeneous core. In this case, we consider the rotation of a circular composite cylinder which is composed of homogeneous material; and non-homogeneous density of non-homogeneity as given in equation (4); up to radius $r = a$, the region bounded by the radii $r = a$ and $r = b$ $(b > a)$ being consisted of non-homogeneous material and density of non-homogenity as given in equations (3) and (4), respectively, for $m = \frac{2}{3}$. Let us denote the region $0 \le r \le a$ by region I and the region $a \le r \le b$ by region II.

In this case, the boundary conditions are:

$$\begin{aligned}
&\text{(i)}\ (\tau_{rr})_{II} = 0 \quad \text{on } r = b,\\
&\text{(ii)}\ \Big[(\tau_{rr})_I\Big]_{r=a} = \Big[(\tau_{rr})_{II}\Big]_{r=a},\\
&\text{(iii)}\ \Big[(u)_I\Big]_{r=a} = \Big[(u)_{II}\Big]_{r=a},\\
&\text{(iv)}\ \Big[(\omega)_I\Big]_{r=a} = \Big[(\omega)_{II}\Big]_{r=a}.
\end{aligned} \tag{15}$$

From the condition (iv) of (15), we get $\epsilon = \epsilon_1$. Again the uniform longitudinal extension ϵ can be so adjusted that the traction on the ends of the cylinder have no statistical result, i.e.,

$$\int_o^a (\tau_{zz})_I\, r\, dr + \int_a^b (\tau_{zz})_{II}\, r\, dr = 0. \tag{16}$$

From the equations (15), (16), (7), (8), (10), (11), (12) and (13), we get

$$A = \frac{k_1 g_2(a) - k_6 f_2(b)}{f_1(b)g_2(a) - f_2(b)g_1(a)},$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$B = \frac{k_6 f_1(b) - k_1 g_1(a)}{f_1(b)g_2(a) - f_2(b)g_1(a)},$$
$$L = \frac{a^2\theta_1 + 4k_5\theta_2}{a^2(a^2 - 4k_5)\Big[f_1(b)g_2(a) - f_2(b)g_1(a)\Big]}, \tag{17}$$
$$\epsilon = \frac{4(\theta_1 + \theta_2)}{(4k_5 - a^2)\Big[f_1(b)g_2(a) - f_2(b)g_1(a)\Big]},$$

where,

$$g_1(a) = f_1(a) - h_1(a), \quad g_2(a) = f_2(a) - h_2(a),$$
$$f_1(r) = kr^{2/3} - \frac{1}{2}k^2 r^{4/3} - 1, \quad f_2(r) = -e^{-kr^{2/3}}, \quad f_3 = k(b^{2/3} - a^{2/3}) - \frac{1}{2}k^2(b^{4/3} - a^{4/3}),$$
$$h_1(r) = (2 - kr^{2/3})e^{kr^{2/3}}, \quad h_2(r) = 2 + kr^{2/3}, \quad h_3 = e^{-ka^{2/3}} - e^{-kb^{2/3}}$$
$$k_1 = \frac{3\rho_o\Omega^2 b^2}{16\mu_o k}(2 + kb^{2/3}), \quad k_2 = \frac{3\rho_o\Omega^2 a^2}{32\mu_o k}(4 - ka^{2/3}), \quad k_3 = \frac{3\rho_o\Omega^2 a^2}{16\mu_o k}(2e^{ka^{2/3}} - ka^{2/3}),$$
$$k_4 = \frac{3\rho_o\Omega^2}{32\mu_o k}\Big[4(b^2 - a^2) + k(b^{8/3} + a^{8/3})\Big],$$
$$k_5 = \frac{15}{4k^3}\Big[\frac{2}{5}a^2k^3 - (k^2b^{4/3} + 2kb^{2/3} + 2)e^{-kb^{2/3}} + (k^2a^{4/3} + 2ka^{2/3} + 2)e^{-ka^{2/3}}\Big],$$
$$k_6 = k_2 - k_3,$$
$$\theta_1 = f_1(b)\Big[k_4 g_2(a) - k_6 h_3\Big] - f_2(b)\Big[k_4 g_1(a) - k_6 f_3\Big] + k_1\Big[g_1(a)h_3 - f_3 g_2(a)\Big],$$
$$\theta_2 = f_1(b)\Big[k_3 g_2(a) - k_6 h_2(a)\Big] - f_2(b)\Big[k_3 g_1(a) - k_6 h_1(a)\Big] + k_1\Big[g_1(a)h_2(a) - g_2(a)h_1(a)\Big].$$

Having obtained the constants from (17), the stresses in the region II ($a \le r \le b$) can be calculated from (8), (9) and (10) and the stresses in the region $I(0 \le r \le a)$ from (12), (13) and (14).

In the associated homogeneous problem, that is, when the whole cylinder is composed of homogeneous material, but of non-homogeneous density, the boundary conditions are given by

$$\tau_{rr} = 0, \quad \text{on } r = b \tag{18}$$

and since ϵ_1 is so adjusted that the traction on the ends of the cylinder has no statstical result, so,

$$\int_o^b \tau_{rr}\, r\, dr = 0. \tag{19}$$

From (18), (19), (12) and (14), the constants L and ϵ_1 are given by

$$L = \frac{3\rho_o\Omega^2}{10\mu_o}b^{2/3}, \quad \epsilon_1 = \frac{-3\rho_o\Omega^2}{40\mu_o}b^{2/3}, \tag{20}$$

also $M = 0$ (since the cylinder is complete).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The stresses in the homogeneous case of material are obtained from (20), (12) and (13).

4. Rotation of a circular cylinder containing a rigid core. Consider the rotation of a circular of radius $r = b$ containing a rigid core of radius $r = a$ having the same axis with the cylinder. In this case, the material and density of the cylinder are non-homogeneous, the non-homogeneity being given in (3) and (4), respectively, for $m = \frac{2}{3}$.

In this case, the boundary conditions are:

$$\text{(i) } u = 0 \text{ on } r = a, \quad \text{(ii) } \tau_{rr} = 0 \text{ on } r = b. \tag{21}$$

Now, we adjusted the uniform longitudinal extension ϵ in such a way that the traction on the ends of the cylinder will have no statistical resultant, and then

$$\int_a^b \tau_{zz}\, r\, dr = 0. \tag{22}$$

From (21), (22), (7), (8) and (10), the constants are given as

$$\begin{aligned}
A &= \frac{K_1\left[4I_2h_2(a) + a^2h_3\right] - f_2(b)\left[4I_1I_2 + a^2k_4\right]}{f_1(b)\left[4I_2h_2(a) + a^2h_3\right] - f_2(b)\left[4I_2h_1(a) + a^2f_3\right]} \\
B &= \frac{f_1(b)\left[4I_1I_2 + a^2k_4\right] - k_1\left[4I_2h_1(a) + a^2f_3\right]}{f_1(b)\left[4I_2h_2(a) + a^2h_3\right] - f_2(b)\left[4I_2h_1(a) + a^2f_3\right]}, \\
\epsilon &= \frac{4\left\{f_1(b)\left[k_4h_2(a) - I_1h_3\right] + k_1\left[h_3h_1(a) - f_3h_2(a)\right] - f_2(b)\left[k_4h_1(a) - I_1f_2\right]\right\}}{f_1(b)\left[4I_2h_2(a) + a^2h_3\right] - f_2(b)\left[4I_2h_1(a) + a^2f_3\right]},
\end{aligned} \tag{23}$$

where, $I_1 = \dfrac{3\rho_0\Omega^2a^2}{8\mu_0 k}e^{ka^{2/3}}$, $I_2 = k_5 - \dfrac{3}{2}a^2$.

The stresses can be calculated from (23), (8), (9) and (10).

In the corresponding homogeneous case of material, from (21), (22), (11), (12) and (13), the constants are given by

$$\begin{aligned}
L &= \frac{\rho_0\Omega^2}{\mu_0}\left[\frac{3}{16}a^{2/3} - \frac{3b^2}{16(b^2-a^2)(3a^2+5b^2)}\left[(b^{8/3}-a^{8/3}) - (b^2-a^2)(9b^{2/3}-5a^{2/3})\right]\right], \\
M &= \frac{3\rho_0\Omega^2}{16\mu_0}a^{8/3} - a^2L, \\
\epsilon &= \frac{\rho_0\Omega^2}{8\mu_0}\cdot\frac{(b^{8/3}-a^{8/3})}{b^2-a^2} - \frac{2}{3}L.
\end{aligned} \tag{24}$$

From (24), (12), (13) and (14) the corresponding stresses can be calculated.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

5. Numerical result and discussion. To give some idea about the effect of non-homogeneity. Let us put

$$P = \frac{\tau_{rr}}{\rho\Omega^2 a^2} \quad \text{and} \quad Q = \frac{\tau_{\theta\theta}}{\rho\Omega^2 a^2}.$$

Now let us suppose $ka^{2/3} = n$, where n is any real number, $\frac{b}{a} = 2$ and $\frac{r}{a} = \delta$.

In the case of the rotation of a circular cylinder containing a homogeneous core, the values of P and Q for the case $n = 1$ are calculated from the equations (12), (13) and (18) in the region I $(0 \leq \delta \leq 1)$ and from the equations (8), (9) and (17) in the region II $(1 \leq \delta \leq 2)$ and they plotted in Fig. 1 and Fig. 2, respectively. In both figures, P_H and Q_H denote the corresponding case when the cylinder material is completely homogeneous.

From Fig. 2, it is observed that for the non-homogeneous case containing a homogeneous core, Q has a certain jump on the interface $\delta = 1$. The amount of jump $Q_c = (Q_I - Q_{II})$ for different values of n is shown in table 1. This table shows that the amount of jump increases with the increases of n. Again in Table 2, we have shown the values of the longitudinal extension

$$E = \frac{-\mu_0 \epsilon}{\rho_0 \Omega^2 a^{2/3}}$$

for different values of n and for the homogeneous case of the material.

For the rotation of a cylinder containing a rigid core, the values of P and Q are calculated for the non-homogeneous case of the material and density by (23) , (8) and (9). Using (24), (12) and (13), the values of P and Q for the homogeneous case have been obtained for different values of $\delta, 1 \leq \delta \leq 2$, which are shown in Fig. 1 and Fig. 2, respectively, where P_H and Q_H are denote the homogeneous case of material.

Finally, Table 3 shows that values of E for different n in the homogeneous case.

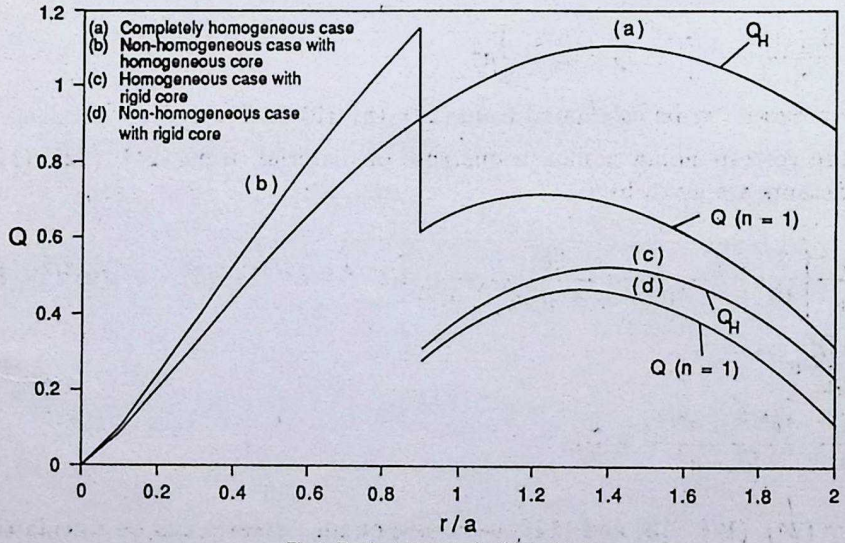


Fig 1 Radical stresses in different cases

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

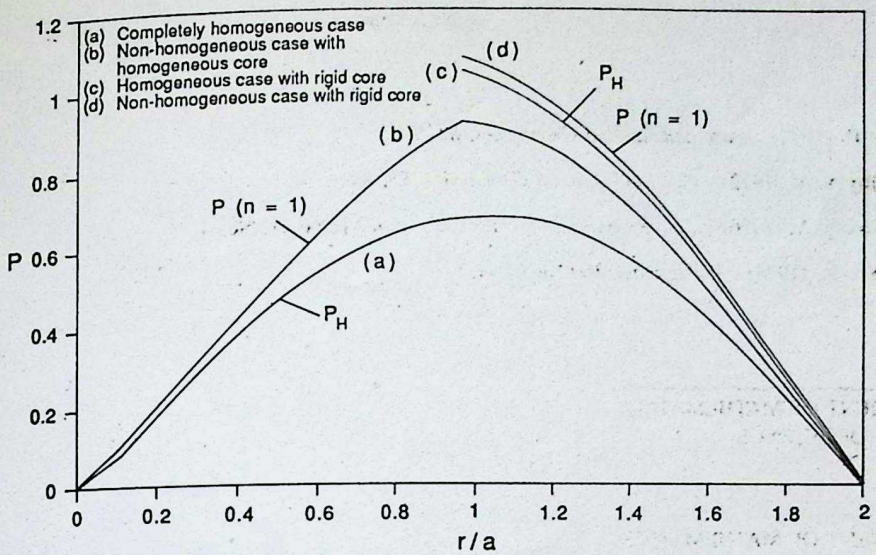


Fig 2 Shearing stresses in different cases

Table 1 Values of $Q_c = (Q_1 - Q_{II})$ for different n

n	1	2	3
Q_c	0.5395	−2.1308	−9.2649

Table 2 Values of E for different n

n	H	1	2	3
E	0.1191	0.3969	0.6841	0.7851

Table 3 Values of E for different n

n	H	1	2	3
E	0.0652	0.3467	1.7343	6.2655

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

References

Chatterjee, D. (1970) : *Pure and Applied Geophysics, 82,* V.

Mukhopadhyay, A. (1972) : *Pure and Applied Geophysics, 93,* VI.

Mukhopadhyay, A. (1976) : *Rev. Roum. Sci. Techn. Meć. Appl.,* Tome 21, No. 4.

Senitskii, Yu. É. (1978) : *Prikladnaya Mekhanika,14,* 5.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
BENHA
EGYPT

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
SOHAG
EGYPT

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
ASWAN
EGYPT

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc., 86, 213–220 (1994)

ON A SEMI-SYMMETRIC NON-METRIC CONNEXION IN ALMOST GRAYAN MANIFOLDS

RAM HIT OJHA AND BHAGWAT PRASAD

(*Received 21 January 1993*)

Abstract. Recently, Nirmala and Mangla (1992) has introduced the semi symmetric non-metric connexion in a Riemannian manifold. In the present paper, we initiate the study of semi-symmetric non-metric connexion in almost Grayan manifolds.

1. Introduction. Let M be an n-dimensional C^∞-manifold and let there exist in M a vector valued linear function F, a vector field T, and 1-form A such that

$$\bar{\bar{X}} + X = A(X)T, \tag{1.1/a}$$

$$\bar{X} \overset{\text{def}}{=} FX, \tag{1.1/b}$$

for arbitrary vector field X. Then M is called an almost contact manifold and the structure (F, T, A) is called an almost contact structure (Sasaki 1960). It follows from (1.1) that the following hold in M: rank $(F) = n-1, n$ is odd, i.e., $n = 2m+1$, say, and

$$\bar{T} = 0, \tag{1.2/a}$$

$$A(\bar{X}) = 0, \tag{1.2/b}$$

$$A(T) = 1. \tag{1.2/c}$$

In addition, if M there exists a metric tensor g satisfying $g(\bar{X}, \bar{Y}) = -g(\bar{\bar{X}}, Y)$ and $(X, T) = A(X)$, which is equivalent to $g(\bar{X}, \bar{Y}) = g(X, Y) - A(X)A(Y)$, then M is called an almost Grayan manifold and (F, T, A, g) an almost Grayan structure (Sasaki 1960).

If D is the Riemannian connection in an almost Grayan manifold, then (Mishra 1972).

$$(D_X{}'F)(\bar{Y}, \bar{\bar{Z}}) = (D_X{}'F)(\bar{\bar{Y}}, \bar{Z}), \tag{1.3/a}$$

$$(D_X{}'F)(\bar{Y}, \bar{Z}) + (D_X{}'F)(\bar{\bar{Y}}, \bar{\bar{Z}}) = 0, \tag{1.3/b}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

and further that in an almost Grayan manifold the Nijenhuis tensor N satisfies

$$N(X,Y) = (D_{\bar{X}}F)(Y) - (D_{\bar{Y}}F)(X) - \overline{(D_X F)(Y)} + \overline{(D_Y F)(X)} \tag{1.4}$$

whence

$$'N(X,Y,Z) = (D_{\bar{X}}\,'F)(Y,Z) - (D_{\bar{Y}}\,'F)(X,Z) + (D_X\,'F)(Y,\bar{Z}) - (D_Y\,'F)(X,\bar{Z}), \tag{1.4}$$

where $'N(X,Y,Z) \stackrel{\text{def}}{=} g(N(X,Y),Z)$.

If in an almost Grayan manifold

$$'F(X,Y) = (D_X A)(Y) - (D_Y A(X) = dA(X,Y), \tag{1}$$

where D denotes the Riemannian connexion, then M is called an almost Sasakian manifo and for an almost Sasakian manifold, we have

$$(D_X\,'F)(Y,Z) + (D_Y\,'F)(Z,X) + (D_Z\,'F)(X,Y) = 0. \tag{1}$$

An almost Sasakian manifold is said to be K-contact Riemannian manifold if

$$'F(X,Y) = (D_X A)(Y) \Longleftrightarrow D_X T = \bar{X}. \tag{1}$$

Hence in a K-contact Riemannian manifold,

$$'F(X,Y) = 2(D_X A)(Y) = -2(D_Y A)(X). \tag{1}$$

An almost contact metric manifold satisfying (Mishra 1991),

$$(D_X\,'F)(Y,Z) = A(Y)(D_X A)(\bar{Z}) - A(Z)(D_X A)(\bar{Y}), \tag{1}$$

is called a generalised cosymplectic manifold.

2. Semi-symmetric non-metric connexion. Let D be the Riemannian connexi in an almost contact manifold M. We consider another affine connexion B in M defined (Nirmala and Mangla 1992)

$$B_X Y = D_X Y + A(Y)X. \tag{2}$$

The connexion B is called a semi-symmetric non-metric connexion. The tersion ten S of B is given by

$$S(X,Y) = A(Y)X - A(X)Y. \tag{2}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Consequently,

$$'S(X,Y,Z) \stackrel{\text{def}}{=} g(S(X,Y),Z) = 'H(X,Y,Z) - 'H(Y,X,Z), \tag{2.3}$$

where

$$'H(X,Y,Z) \stackrel{\text{def}}{=} g(H(X,Y),Z). \tag{2.4}$$

Further, we have (Nirmala and Mangla 1992)

$$'H(X,Y,Z) + 'H(X,Z,Y) = A(Y)g(X,Z) + A(Z)g(X,Y), \tag{2.5/a}$$
$$(B_X g)(Y,Z) = -A(Y)g(X,Z) - A(Z)g(X,Y). \tag{2.5/b}$$

From (2.1), in the almost contact manifold, we have the relation

$$(B_X A)(Y) = (D_X A)(Y) - A(X)A(Y). \tag{2.6}$$

THEOREM (2.1). *On an almost contact metric manifold with the connexion B, we have*

$$(B_X 'F)(Y,Z) = (D_X 'F)(Y,Z) - A(Y)'F(X,Z) - A(Z)'F(Y,X), \tag{2.7}$$
$$g((B_X F)Y,Z) = g((D_X F)Y,Z) - A(Y)g(\bar{X},Z). \tag{2.8}$$

Proof. We have

$$(B_X 'F)(Y,Z) = X'F(Y,Z) - 'F(B_X Y,Z) - 'F(Y,B_X Z), \tag{2.9}$$

using (2.1) in (2.9), we get (2.7).

Finally, to establish (2.8), we have

$$\begin{aligned} g((B_X F)Y,Z) &= g(B_X \bar{Y},Z) - g(\overline{B_X Y},Z) \\ &= g(D_X \bar{Y},Z) - g(\overline{D_X Y} + A(Y)\bar{X},Z) \\ &= g((D_X F)Y,Z) - A(Y)g(\bar{X},Z). \end{aligned}$$

THEOREM (2.2). *On an almost contact metric manifold with the connexion B, we have*

$$(B_X 'F)(\bar{Y},\bar{\bar{Z}}) = (B_X 'F)(\bar{\bar{Y}},\bar{Z}), \tag{2.10/a}$$
$$(d'F)(\bar{X},\bar{Y},\bar{Z}) = (B_{\bar{X}} 'F)(\bar{Y},\bar{Z}) + (B_{\bar{Y}} 'F)(\bar{Z},\bar{X}) + (B_{\bar{Z}} 'F)(\bar{X},\bar{Y}). \tag{2.10/b}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. Using (1.3/a) and (2.7), we get (2.10/a).

Further, we have from (2.7)

$$(B_{\bar{X}}\,'F)(\bar{Y},\bar{Z}) = (D_{\bar{X}}\,'F)(\bar{Y},\bar{Z}). \tag{2.11}$$

Writing two other terms in cyclic order, and then adding, we obtain (2.10/b).

THEOREM (2.3). *An almost contact metric manifold with the semi-symmetric non-metric connexion B is an almost Sasakian manifold iff*

$$'F(X,Y) = (B_X A)(Y) - (B_Y A)(X). \tag{2.12}$$

Proof. From (2.6), we obtain

$$(B_X A)(Y) - (B_Y A)(X) = (D_X A)(Y) - (D_Y A)(X),$$

which proves the statement.

THEOREM (2.4). *On a K-contact Riemannian manifold with the connexion B, we have*

$$A((B_X F)(Y)) = -\frac{1}{2}g(\bar{X},\bar{Y}), \tag{2.13/a}$$

$$(B_T\,'F)(\bar{Y},\bar{Z}) + (B_T\,'F)(Y,Z) = 0. \tag{2.13/b}$$

Proof. By using (1.8) and (2.6), we have

$$(B_X A)(Y) = \frac{1}{2}\,'F(X,Y) - A(X)A(Y). \tag{2.14}$$

In consequence of (2.14), we have

$$\begin{aligned} A((B_X F)Y) = A(B_X\bar{Y}) = -(B_X A)(\bar{Y}) &= -\frac{1}{2}\,'F(X,\bar{Y}) \\ &= -\frac{1}{2}g(\bar{X},\bar{Y}). \end{aligned}$$

Hence (2.13/a) is proved. For (2.13/b), we have

$$\begin{aligned} &(B_X\,'F)(\bar{Y},\bar{Z}) + \,'F(B_X\bar{Y},\bar{Z}) + \,'F(\bar{Y},B_X\bar{Z}) \\ &\quad = (B_X\,'F)(Y,Z) + \,'F(B_X Y,Z) + \,'F(Y,B_X Z). \end{aligned} \tag{2.15}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Now,

$$
\begin{aligned}
{}'F(B_X\bar{Y},\bar{Z}) &= g(\overline{B_X\bar{Y}},\bar{Z}) = g(B_X\bar{Y},Z) - A(B_X\bar{Y})A(Z) \\
&= g((B_XF)Y,Z) + g(\overline{B_XY},Z) - A(Z)A((B_XF)Y).
\end{aligned} \tag{2.16}
$$

Again, we have

$$
\begin{aligned}
X\,{}'F(Y,Z) &= (B_X\,{}'F)(Y,Z) + {}'F(B_X,Y,Z) + {}'F(Y,B_XZ) \\
&= (B_Xg)(\bar{Y},Z) + g(B_X\bar{Y},Z) + g(\bar{Y},B_XZ) \\
&= -A(Z)g(X,\bar{Y}) + g(B_X\bar{Y},Z) + g(\bar{Y},B_XZ)
\end{aligned}
$$

which is equivalent to

$$
(B_X\,{}'F)(Y,Z) = g((B_XF)Y,Z) - A(Z)\,{}'F(X,Y). \tag{2.17}
$$

Thus, in consequence of (2.15), (2.16) and (2.17), we obtain

$$
\begin{aligned}
&(B_X\,{}'F)(\bar{Y},\bar{Z}) + (B_X\,{}'F)(Y,Z) \\
&\quad = \frac{1}{2}\{A(Y)g(X,Z) - A(Z)g(X,Y)\} + A(Z)\,{}'F(Y,X) + A(Y)\,{}'F(X,Z).
\end{aligned} \tag{2.18}
$$

Putting $X = T$ in (2.18), we get (2.13/b).

THEOREM (2.5). *Let B be a semi-symmetric non-metric connexion. Then, we have*

$$
{}'S(\bar{X},Y,\bar{Z}) + {}'S(Y,\bar{Z},\bar{X}) - {}'S(\bar{Z},\bar{X},Y) = {}'H(Y,\bar{Z},\bar{X}) - {}'H(Y,\bar{X},\bar{Z}). \tag{2.19}
$$

Proof. From (2.3), we have

$$
{}'S(X,Y,Z) = {}'H(X,Y,Z) - {}'H(Y,X,Z). \tag{2.20}
$$

Similarly, we have

$$
{}'S(Y,Z,X) = {}'H(Y,Z,X) - {}'H(Z,Y,X) \tag{2.21}
$$

$$
{}'S(Z,X,Y) = {}'H(Z,X,Y) - {}'H(X,Z,Y). \tag{2.22}
$$

In consequence of (2.20), (2.21), (2.22) and (2.5/a) we get (2.19).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

THEOREM (2.6). *On generalised co-symplectic manifold, we have*

$$(B_T\,'F)(\bar{Y},\bar{Z}) = 0 \tag{2.23/a}$$
$$(B_{\bar{X}}\,'F)(\bar{Y},\bar{Z}) + (B_{\bar{Y}}\,'F)(\bar{Z},\bar{X}) + (B_{\bar{Z}}\,'F)(\bar{X},\bar{Y}) = 0. \tag{2.23/b}$$

Proof. In consequence of (1.9) and (2.7), we get

$$\begin{aligned}(B_X\,'F)(Y,Z) + A(Y)\,'F(X,Z) + A(Z)\,'F(Y,X)\\ = A(Y)(B_X A)(\bar{Z}) - A(Z)(B_X A)(\bar{Y}).\end{aligned} \tag{2.24}$$

Putting $X = T$ in (2.24), we get (2.23/a). (2.23/b) follows by simple calculation.

THEOREM (2.7). *A generalised co-symplectic manifold is completely integrable with respect to the semi-symmetric non-metric connexion B.*

Proof. Using (1.4/b) and (2.7), we get

$$'N(X,Y,Z) = (B_{\bar{X}}\,'F)(Y,Z) - (B_{\bar{Y}}\,'F)(X,Z) + (B_X\,'F)((Y,\bar{Z}) - (B_Y\,'F)(X,\bar{Z}). \tag{2.25}$$

We have from (2.24)

$$(B_X\,'F))(Y,Z) = A(Y)[(B_X A)(\bar{Z}) - \,'F(X,Z)] - A(Z).[(B_X A)(\bar{Y}) + \,'F(Y,X)] \tag{2.26}$$

In consequence of (2.26) and (2.25), we have

$$'N(\bar{X},\bar{Y},\bar{Z}) = 0$$

which proves the statement.

3. Curvature tensor of a Semi-symmetric non-metric connexion. We denote the curvature tensor of the connexion B by R and the curvature tensor of the connexion D by K. By straight forward calculation, we obtain

$$R(X,Y,Z) = K(X,Y,Z) + ((B_X A)Z)Y - ((B_Y A)Z)X \tag{3.1}$$

Let

$$\mathrm{Ric}(Y,Z) \overset{\text{def}}{=} (C_1^1 K)(Y,Z) \tag{3.2/a}$$
$$\mathrm{Ric}(Y,Z) \overset{\text{def}}{=} (C_1^1 R)(Y,Z) \tag{3.2/b}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Then (3.1) gives

$$\mathrm{Ric}'(Y,Z) = \mathrm{Ric}(Y,Z) - (B_Y A)(Z)(n-1). \tag{3.3}$$

Let us put

$$'R(X,Y,Z,W) \stackrel{\text{def}}{=} g(R(X,Y,Z)'W) \tag{3.4/a}$$

$$\text{and} \quad 'K(X,Y,Z,W) \stackrel{\text{def}}{=} g(K(X,Y,Z)'W). \tag{3.4/b}$$

Then

$$'R(X,Y,Z,W) = 'K(X,Y,Z,W) + (B_X A)(Z)g(Y,W) - (B_Y A)(Z)g(X,W). \tag{3.5}$$

THEOREM (3.1). *On a K-contact Riemannian manifold, we have*

$$'R(X,Y,\bar{Z},\bar{W}) + 'R(\bar{X},\bar{Y},Z,W) = 'K(X,Y,\bar{Z},\bar{W}) + K(\bar{X},\bar{Y},Z,W) + g(\bar{X},W)g(\bar{Y},\bar{Z}) - g(\bar{Y},W)g(\bar{X},\bar{Z}), \tag{3.6/a}$$

$$'R(T,Y,\bar{Z},\bar{W}) + 'R(T,Y,\bar{W},\bar{Z}) = 0, \tag{3.6/b}$$

$$'R(T,Y,Z,\bar{W}) - 'R(Z,\bar{W},T,Y) = 0. \tag{3.6/c}$$

Proof. In consequence of (3.5) and (2.14), we have

$$'R(X,Y,Z,W) = 'K(X,Y,Z,W) + \frac{1}{2}'F(X,Z)g(Y,W) - A(X)A(Z)g(Y,W) + \frac{1}{2}'F(Y,Z)g(X,W) - A(Y)A(Z)g(X,W) \tag{3.7}$$

Baring Z and W in (3.7), we get

$$'R(X,Y,\bar{Z},\bar{W}) = 'K(X,Y,\bar{Z},\bar{W}) + \frac{1}{2}'F(X,\bar{Z})g(Y,\bar{W}) + \frac{1}{2}'F(Y,\bar{Z})g(X,\bar{W}) \tag{3.8}$$

Similarly, we have

$$'R(\bar{X},\bar{Y},Z,W) = 'K(\bar{X},\bar{Y},Z,W) + \frac{1}{2}'F(\bar{X},Z)g(\bar{Y},W) + \frac{1}{2}'F(\bar{Y},Z)g(\bar{X},W). \tag{3.9}$$

Adding (3.8) and (3.9), we get (3.6/a). Similarly other results can also be obtained by direct computation.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

THEOREM (3.2). *On a K-contact manifold, the scalar curvature is given by*

$$r' = r + n - 1. \tag{3.10}$$

Proof. We have $g(T, Z) = A(Z)$

$$(B_Y g)(T, Z) + g(B_Y T, Z) = (B_Y A)(Z). \tag{3.11}$$

Using (2.5/b) in (3.11), we get

$$-g(Y, Z) - g(Z, T)g(Y, T) + g(B_Y T, Z) = (B_Y A)(Z). \tag{3.12}$$

In consequence of (3.3) and (3.12), we obtain

$$R'Y = RY + (Y - A(Y)T - B_Y T)(n - 1). \tag{3.13}$$

Using (1.7) in (3.13) and contracting, we get required result.

References

Mishra, R. S. (1991) : *Almost contact metric manifold Tensor Society of India*, Lucknow, India.

Mishra, R. S. (1972) : Affine connections in an almost Grayan manifold, *Tensor, N. S., 23*, 3.

Nirmala, S., Agashe and **Mangla, R. Chafle** (1992) : A semi-symmetric non-metric connextion on a Riemannian manifold. *India J. pure appl. Math.*, 23(6), 399.

Sasaki, S. (1960) : On differentiable manifolds with certain structures which are closely related to almost contact structure. *Tohoku Math. J.*, 12, 456.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
BANARAS HINDU UNIVERSITY
VARANASI 221 005
INDIA

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc., 86, 221–226 (1994)

ON DECOMPOSITION OF CURVATURE TENSOR FIELDS IN TACHIBANA RECURRENT SPACES

A. K. SINGH AND K. C. PETWAL

(*Received 20 October 1992*)

1. Introduction. An almost Techibana space is first of all an almost Hermite space, that is, a $2n$-dimensional space with almost complex structure F:

$$F_j^i F_i^h = -\delta_j^h \tag{1.1}$$

and with a Riemannian metric g_{ji}, satisfying

Gurukul Kangri Vishwavidyalaya
HARDWAR

$$F_j^a F_i^b g_{ab} = g_{ji}, \tag{1.2}$$

from which

$$F_{ji} = -F_{ij}, \tag{1.3}$$

where

$$F_{ji} = F_j^a g_{ai} \tag{1.4}$$

and finally has the property that the skew-symmetric tensor F_{ih} is a killing tensor:

$$F_{ih,j} + F_{jh,i} = 0, \tag{1.5}$$

from which

$$F_{i,j}^h + F_{j,i}^h = 0, \tag{1.6}$$

and

$$F_i = -F_{i,j}^j = 0. \tag{1.7}$$

Now the Nijenhuis tensor N_{ji}^h is written in the form

$$N_{ji}^h = -4(F_{i,j}^a)F_a^h + 2G_{ji}^a F_a^h + F_j^a G_{ai}^h - F_i^a G_{aj}^h. \tag{1.8}$$

and consequently, in an almost Tachibana space, we have

$$N_{ji}^h = -4(F_{i,j}^a)F_a^h \tag{1.9}$$

where $F_{i,j}^h$, is pure in J and i.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

When the Nijenhuis tensor vanishes, the almost Tachibana space is called a Tachibana space. In this case, we have from (1.9)

$$F^h_{i,j} = 0 \tag{1.10}$$

where the comma (,) followed by an index denotes the operator of covariant differentiation with respect to the metric tensor g_{ji} or the Riemannian space.

The Riemannian curvature tensor field is defined by*

$$R^h_{ijk} = \partial_i \left\{ {h \atop jk} \right\} - \partial_j \left\{ {h \atop ik} \right\} + \left\{ {h \atop il} \right\} \left\{ {l \atop jk} \right\} - \left\{ {h \atop jl} \right\} \left\{ {l \atop ik} \right\} \tag{1.11}$$

the Ricci tensor and scalar curvature are given by $R_{ij} = R^a_{aij}$ and $R = g^{ij} R_{ij}$, respectively.

It is well-known that these tensors satisfy the following identities (Tachibana 1967).

$$R^a_{ijk,a} = R_{jk,i} - R_{ik,j}, \tag{1.12}$$

$$R,i = 2R^a_i, a, \tag{1.13}$$

$$F^a_i R_{aj} = -R_{ia} F^a_j. \tag{1.14}$$

and

$$F^a_i F^j_a = R^a_i F^j_a. \tag{1.15}$$

The Bianchi identities in T^c_n are given by

$$R^h_{ijk} + R^h_{jki} + R^h_{kij} = 0 \tag{1.16}$$

and

$$R^h_{ijk,a} + R^h_{ika,j} + R^h_{iaj,k} = 0. \tag{1.17}$$

A Tachibana space is said to be Tachibana recurrent if its curvature tensor field satisfies the condition

$$R^h_{ijk,a} = \lambda_a R^h_{ijk}, \tag{1.18}$$

where λ_a is a non-zero recurrence vector field.

The following relations follows immediately from equation (1.18)

$$R_{ik,a} = \lambda_a R_{ik} \tag{1.19}$$

and

$$R,a = \lambda_a R \tag{1.20}$$

* $\partial i = \partial/\partial x^i$, where $\{x^i\}$ denotes the real local coordinates.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In the present paper, we have studied the decomposition of curvature tensor field R^h_{ijk} in terms of two non-zero tensor fields and several theorems have been investigated.

2. Decomposition of Curvature Tensor Field $\mathbf{R^h_{ijk}}$. Consider the decomposition of recurrent curvature tensor field R^h_{ijk} in the following form:

$$R^h_{ijk} = A_{jk} v^h, i, \tag{2.1}$$

where A_{jk} and $v^h_{,i}$ are two non-zero tensor fields such that

$$\lambda_h v^h_{,i} = p_i, \tag{2.2}$$

$$\lambda_h v^h = 1 \tag{2.3}$$

$$\text{and} \quad A_{jk} = \lambda_{j,k} - \lambda_{k,j} \neq 0. \tag{2.4}$$

The vector field p_i is also a non-zero vector field and is called decomposed vector field.

We shall prove the following:

THEOREM (2.1). *Under the decomposition* (2.1), *the Bianchi identities take the forms:*

$$p_i A_{jk} + p_j A_{ki} + p_k A_{ij} = 0 \tag{2.5}$$

$$\text{and} \quad \lambda_a A_{jk} + \lambda_j A_{ka} + \lambda_k A_{aj} = 0. \tag{2.6}$$

Proof. In view of equations (1.16) and (2.1), we have

$$A_{jk} v^h_{,i} + A_{ki} v^h_{,j} + A_{ij} v^h_{,k} = 0. \tag{2.7}$$

Multiplying the above equations by λ_h and using (2.2), we obtain the required result (2.5).

Making use of equations (1.17), (1.18) and (2.1), we get

$$v^h_{,i}(\lambda_a A_{jk} + \lambda_j A_{ka} + \lambda_k A_{aj}) = 0. \tag{2.8}$$

Multiplying the above equation by λ_h and using (2.2), we have

$$p_i(\lambda_a A_{jk} + \lambda_i A_{ka} + \lambda_k A_{aj}) = 0. \tag{2.9}$$

Since p_i is a non-zero recurrence vector field, we obtain

$$\lambda_a A_{jk} + \lambda_j A_{ka} + \lambda_k A_{aj} = 0$$

This completes the proof.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

THEOREM (2.2). *Under the decomposition* (2.1), *the tensor fields* R^h_{ijk}, R_{ij} *and* A_{jk} *satisfy the relations*

$$\lambda_a R^a_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik} = p_i A_{jk} \tag{2.10}$$

Proof. With the help of equations (1.12), (1.18) and (1.19), we obtain

$$\lambda_a R^a_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik}. \tag{2.11}$$

Multiplying equation (2.1) by λ_h and using the relation (2.2), we have

$$\lambda_a R^a_{ijk} = p_i A_{jk}. \tag{2.12}$$

In view of equations (2.11) and (2.12), we get the required result (2.10).

THEOREM (2.3). *Under the decomposition* (2.1), *the quantities* λ_a *and* $v^h_{,i}$ *behave like recurrent vector and tensor fields, respectively. The recurrent forms of these quantities are given by*

$$\lambda_{a,m} = \mu_m \lambda_a \tag{2.13}$$

$$\text{and} \quad v^h_{,im} = \nu_m v^h_{,i}, \tag{2.14}$$

where $v^h_{,i,m} \overset{\text{def.}}{=} v^h_{,im}$ *and* μ_m, ν_m *are non-recurrence vector fields.*

Proof. Differentiating (2.11) covariantly with respect to x^m and using equations (2.1) and (2.10), we have

$$\lambda_{a,m} A_{jk} v^a_{,i} = \lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}. \tag{2.15}$$

Multiplying the above equation by λ_a and using (2.11), we obtain

$$\lambda_{a,m}(\lambda_i R_{jk} - \lambda_j R_{ik}) = \lambda_a(\lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}). \tag{2.16}$$

Now, multiplying equation (2.16) by λ_h, we get

$$\lambda_{a,m}(\lambda_i R_{jk} - \lambda_j R_{ik})\lambda_h = \lambda_a \lambda_h(\lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}). \tag{2.17}$$

Since the expression on the right hand side of the above equation is symmetric in 'a' and 'h', therefore we get

$$\lambda_{a,m}\lambda_h = \lambda_{h,m}\lambda_a, \tag{2.18}$$

provided that

$$\lambda_i R_{jk} - \lambda_j R_{ik} \neq 0.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The vector λ_a being non-zero, we can obtain a proportional vector field μ_m such that

$$\lambda_{a,m} = \mu_m \lambda_a.$$

Further, differentiating (2.2) covariantly with respect to x^m and using (2.13), we get

$$\lambda_h v^h_{,im} = p_{i,m} - \mu_m p_i. \tag{2.19}$$

From the above equation, it is obvious that

$$\lambda_h v^h_{,im} = \lambda_a v^a_{,im} \tag{2.20}$$

Since λ_a is a non-zero recurrence vector field, we can get a proportional vector field ν_m such that

$$v^h_{,im} = \nu_m v^h_{,i},$$

This completes the proof.

THEOREM (2.4). *Under the decomposition* (2.1), *the vector field* p_i *and tensor field* A_{jk} *behave like recurrent vector and recurrent tensor fields and their recurrent forms are given respectively by*

$$p_{i,m} = (\mu_m + \nu_m)p_i \tag{2.21}$$

$$\text{and} \quad A_{jk,m} = (\lambda_m - \nu_m)A_{jk}. \tag{2.22}$$

Proof. Differentiating (2.2) covariantly with respect to x^m and using equations (2.2), (2.13) and (2.14), we obtain the required result (2.21).

Further, differentiating equation (2.1) covariantly with respect to x^m and using equations (1.18), (2.1) and (2.14), we get the required recurrent form (2.22).

THEOREM (2.5). *Under the decomposition* (2.1), *the curvature tensor and holomorphically projective curvature tensor are equal iff*

$$A_{km}\left\{(p_i\delta^h_j - p_j\delta^h_i) + p_a(F^h_j F^a_i - F^h_i F^a_j)\right\} + 2p_a A_{jln} F^h_k F^a_i = 0. \tag{2.23}$$

Proof. The holomorphically projective curvature tensor field P^h_{ijk} in K^c_n is defined by

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{n+2}(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - \delta_{jk}F^h_i + 2S_{ij}F^h_k), \tag{2.24}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

which may be expressed briefly as

$$P^h_{ijk} = R^h_{ijk} + D^h_{ijk}, \tag{2.25}$$

$$\text{where} \quad D^h_{ijk} = \frac{1}{n+2}\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}F^h_i + 2S_{ij}F^h_k\right). \tag{2.26}$$

Contracting indices 'h' and 'k' in (2.1), we obtain

$$R_{ij} = A_{jl}v^l_{,i}. \tag{2.27}$$

In view of the above equation, we have

$$S_{ij} = F^a_i R_{aj} = F^a_i A_{jl} v^l_{,a}. \tag{2.28}$$

Making use of (2.27) and (2.28) in (2.26), we have

$$D^h_{ijk} = \frac{1}{n+2}\left[A_{kl}\{(\delta^h_j v^l_{,i} - \delta^h_i v^l_{,j}) + v^l_{,a}(F^h_j F^a_i - F^h_i F^a_j) + 2A_{jl}F^a_i F^h_k v^l_{,a}\right]. \tag{2.29}$$

From (2.25) it is clear that

$$P^h_{ijk} = R^h_{ijk} \text{ iff } D^h_{ijk} = 0,$$

which in equation (2.29) gives

$$A_{kl}\{(\delta^h_j v^l_{,a} - \delta^h_i v^l_{,j}) + v^l_{,a}(F^h_j F^a_i - F^h_i F^a_j)\} + 2A_{jl}F^a_i F^h_k v^l_{,a} = 0. \tag{2.30}$$

Multiplying the above equation by λ_a and using (2.2), we obtain the required condition (2.23).

References

Lal, K. B. and **Singh, S. S.** (1970) : On Kaehlerian spaces with recurrent Bochner curvature, *Acc. Naz. Dei Lincei, Rend Series VIII, LI*, 34, 91.

Takano, K. (1967) : Decomposition of curvature tensor in a recurrent space, *Tensor* (*N.S.*),*18*, 3, 343.

Tachibana, S. (1967) : On the Bochner curvature tensor, *Net. Sci. Rep., Ochanomuzu University, 18*, 1, 15.

Yano, K. (1965) : *Differential geometry on complex and almost complex spaces*, Pergamon Press.

DEPARTMENT OF MATHEMATICS
H.N.B. GARHWAL UNIVERSITY
CAMPUS TEHRI
TEHRI GARHWAL 249001 (U.P.)
INDIA

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

ll. Cal. Math. Soc., 86, 227–232 (1994)

GENERALISED SIMPLE EIGENVALUES AND BIFURCATION FOR A LINKED MULTIPARAMETER EIGENVALUE PROBLEM

M. H. SALLAM

(*Received 19 October 1992*)

1. Introduction. Suppose that we are given a multiparameter system of equation,

$$L_i(\underline{\lambda})x_i = F_i(\underline{\lambda}, x_i, \ldots, x_m); \tag{1.1}$$

$$L_i(\underline{\lambda}) = A_i - \sum_{j=1}^{n} \lambda_j B_{ij}, i = 1, \ldots, m; m \leq n,$$

ere A_i, B_{ij} are bounded self-adjoint operators on Hilbert spaces $H_i, i = 1, \ldots, m$ and $\lambda_j, j =$ $\ldots, n$ are real parameters. An eigenvalue of (1.1) is a point $\underline{\lambda} = (\lambda_1 \ldots, \lambda_n)$ for which each ation possess a solution $x_i \neq 0$. The spectral theory of multiparameter system of the case $= n$ has been considered in many recent papers, (Binding 1984), (Browne and Sleeman 1) for abstract problem and in (Browne 1977, 1979, 1980) for a linked system of non-linear nd order ordinary differential equations. In this paper we are concerned with the problem ifurcation of solutions of the non-linear problem (1.1) at a generalised simple eigenvalue he linearised problem for the case $m \leq n$. The result is obtained using the concept of eralised simple eigenvalues that we have introduced in (McGhee and Sallam 1988). Using concept we also investigate the multiparameter eigenvalue problem in the case where some he operators depend analytically on a perturbation parameter. The distribution of this er is as follows: In section 2 we give our definition for the generalised simple eigenvalue. In ion 3, Theorem 3.2, show that, under some standard conditions on the nonlinear terms, (1.1) a set of solutions bifurcating from the trivial solution $(\underline{\lambda}, 0) \in \mathbb{R}^n x X$ at such generalised ple eigenvalues. Theorem 3.3 gives the same results when some of the operators depend lytically on perturbation parameter.

2. Definition of a generalised simple eigenvalue. Let x, y be real Banach spaces let $A, B_i, i = 1, \ldots, n$ be bounded linear operators from x into y. Consider the following blem

$$L(\underline{\lambda})x + N(\underline{\lambda}, x) = 0, \tag{2.1}$$

re

$$L(\underline{\lambda}) = A - \sum_{j=1}^{n} \lambda_j B_j \tag{2.2}$$

$\underline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In (McGhee and Sallam 1988) we have introduced the following definition of a generalised simple eigenvalue.

DEFINITION 2.1. *$\underline{\lambda}^{\circ} = (\lambda_1^{\circ}, \ldots, \lambda_n^{\circ}) \in \mathbb{R}^n$ is a generalised simple eigenvalue $(A, B_1, \ldots, B_n)$ if;*

(i) $\dim N(L(\underline{\lambda}^{\circ}) = 1$;

(ii) *$L(\underline{\lambda}^{\circ})$ is a Fredholm operator of index $1 - m$ where $m \leq n$;*

(iii) *$B_i x_0 \notin R(1(\underline{\lambda}^{\circ})), i = 1, \ldots, n$ where $x_0 \in N(L(\underline{\lambda}^{\circ}))$ and $Y = span\ \{B_i x_0, i = 1, \ldots, n) \oplus R(L(\underline{\lambda}^{\circ}))$.*

By using Definition 2.1. we have proved in (McGhee and Sallam 1988) the following result.

THEOREM 2.2. *Let $\underline{\lambda}^{\circ} \in \mathbb{R}^n$ be a generalised simple eigenvalue of (2.1) and $N : \mathbb{R}^n \times x \longrightarrow y$ be a non-linear mapping such that*

$$C1 : N \in C^r(\mathbb{R}^n \times x, y), r \geq 2;$$
$$C2 : N(\underline{\lambda}, 0) = 0;$$
$$C3 : D_x N(\underline{\lambda} m, \underline{\mu}\ \overset{\circ}{m}), 0 = 0,$$

where $\underline{\lambda} m = (\lambda_1, \ldots, \lambda_m), \mu = (\underline{\lambda}_m, \ldots, \lambda_{m+1})$.

Then $(\underline{\lambda}^{\circ}, o) \in \mathbb{R}^n \times X$ is a bifurcation point of solutions of (2.1) and there exist set of solutions.

$$\{(\underline{\lambda}, x) = ((\underline{\lambda}_m^*(u, \underline{\mu}_m), \underline{\mu}_m), x^*(u, \underline{\mu}_m) : \\ u \in (-\delta.\delta) \subset \mathbb{R} \quad \text{for some } \delta > 0; \\ \|\underline{\mu}_m - \underline{\mu}_m\| < \epsilon \text{ for some } \epsilon > 0\},$$

where $\underline{\lambda}_m^* : \mathbb{R} \times \mathbb{R}^{n-m} \longrightarrow \mathbb{R}^m$ and $x^* : \mathbb{R} * \mathbb{R}^{n-m} \longrightarrow X$ are C^{r-1} mapping.

We shall use the results of Theorem 2.2 to study the linked system (1.1).

3. Linked Multiparameter Eigenvalue Problems. Consider the system

$$M_i(\underline{\lambda}, x) := \left(A_i \sum_{j=1}^{n} \lambda_j B_{ij}\right) x_i - f_i(\underline{\lambda}, x) = 0,$$

$i = 1, \ldots, m; m \leq n$ where $x = (x_1, \ldots, x_m), x_i \in H_i, A_i, B_{ij}$ are all Hermitian (bounded self-adjoint linear operators) and $f_i : \mathbb{R}^n \times H_i \longrightarrow H_i$ is a non-linear mapping.

Define $x = x_{i=1}^m H_i$ which is a linear product space when given the inner product.

$$(x, y) = \sum_{i=1}^{m} (x_i, y_i)_{H_i}, x_i, y_i \in H_i.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

It is clear that x with this inner product becomes a Hilbert space. Follow Binding 1984) (see also McGhee and Sallam 1988) we define the following operators:

$$A : X \longrightarrow X; \quad B_j : X \longrightarrow X, J = 1, \ldots, n,$$

here $Ax = (A_1, x_1, \ldots, A_m x_m); B_j x = (B_{1j} x_1, \ldots, B_{mj} x_m)$; and $f(\underline{\lambda}, x) = (f_1(\underline{\lambda}; x), \ldots, {}_m(\underline{\lambda}; x))$.

The system (3.1) is then equivalent to the single problem

$$M(\underline{\lambda}; x); = w(\underline{\lambda})x - f(\underline{\lambda}; x) = 0, \tag{3.2}$$

here

$$w(\underline{\lambda}); = A - \sum_{i=1}^{n} \lambda_j B_j. \tag{3.3}$$

With these notations, we obtain the following result:

LEMMA 3.1. *Assuming that for $\underline{\lambda} = \underline{\lambda}^\circ \in \mathbb{R}^n$ each of the linear problem in the system 3.1) has exactly one linear independent solution $x_i^\circ, i = 1, \ldots, m; R(L_i(\underline{\lambda}))$ is closed and the atrix $S(x^\circ) = S(x_1^\circ, \ldots, x_m^\circ) = ((B_{ij} x_i^\circ, x_i^\circ)), i = 1, \ldots, m; j = 1, \ldots, n$ is such that all the terminations of order m not equal to zero. Then the equivalent equation*

$$w(\underline{\lambda})x = \left(A - \sum_{j=1}^{n} \lambda_j B_j\right) x = 0$$

tisfy the following:

(i) $w(\underline{\lambda})$ is a self-adjoint operator for all $\underline{\lambda} \in \mathbb{R}^n$

(ii) $w(\underline{\lambda}^\circ)$ *is a Fredholm operator with zero index; dim* $n(L(\lambda^\circ)) = m$ *where* $N(L(\lambda^\circ)) = spen[\xi_1^\circ, \ldots, \xi_m^\circ], \xi_i^\circ = (0, \ldots, 0, x_i^\circ, 0 \ldots, 0)$ and $y_i = R(L(\underline{\lambda}^\circ)) = [spen[\xi_1^\circ, \ldots, \xi_m^\circ]]^\perp$

(iii) $B_j x^\circ \notin y_1; x^\circ = (x_1^\circ, \ldots, x_m^\circ), j = 1, \ldots, n$ *and dim* $y_\circ = m \leq n$ *where* $y_\circ = spen[B_1 x^\circ, \ldots, B_n x^\circ]$

Hence,

$$x = y_\circ \oplus y_1. \tag{3.4}$$

Proof. (i) and (ii) are easy and we shall prove (iii) since all the determinants of order m t equal to zero, then for each j there exist $(B_{i,j} x_i^\circ, x_i^\circ) \neq 0 \longrightarrow (B_j x^\circ, \xi_i^\circ) \neq 0 \longrightarrow B_j x^\circ \neq y_1$.

Also, since rank $S(x^\circ) = m \leq n$, then $dim\ y_\circ = m$ and hence: $x = y_\circ \oplus y_1$.

This lemma is exactly the result for the case $m = n$

We also not that x has the following direct sum

$$X = X_\circ \oplus X_1, \tag{3.5}$$

ere $X_\circ - spen(x^\circ)$ and $X_1 = [\text{spen } [\xi_1, \ldots, \xi_m^\circ]]^\perp$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

For the non-linear equation (3.2) we have the following:

THEOREM 3.2. *Let* $\underline{\lambda}^{\circ} \in \mathbb{R}$ *be as in lemma 3.1 and let* $f_i;\ \mathbb{R}^n \times X \longrightarrow H_i, i = 1,..$ *satisfy:*

$$C1 f_i \in C^r(\mathbb{R}^n \times X, H_i), r \geq 2$$
$$C2 f_i(\underline{\lambda}, 0) = 0 \forall \underline{\lambda} \in \mathbb{R}^n$$
$$C3 D_x f_i((\underline{\lambda} m, \underline{\mu}^{\circ} \overset{\circ}{m})0) = 0$$

then $(\underline{\lambda}^{\circ}, 0) \in \mathbb{R}^n \times X$ *is a bifurcation point for solutions of (3.2) and there exists a s solutions*

$$\{(\underline{\lambda}, x) = (\underline{\lambda}^* m(u, \underline{\mu} m), \underline{\mu} m); x_1^*(u, \underline{\mu} m), \ldots, x_m^*));$$
$$u \in (-\delta, \delta) \subset \mathbb{R} \text{ for some } \delta > 0;$$
$$\|\underline{\mu} m - \underline{\mu} \overset{\circ}{m}\| < \epsilon \text{ for some } \epsilon > 0\},$$

where $\underline{\lambda}^* m;\ \mathbb{R} \times \mathbb{R}^{n-m} \longrightarrow \mathbb{R}^m$ and $x_i^*;\ \mathbb{R} \times \mathbb{R}^{n-m} \longrightarrow H_i, i = 1, \ldots, m$ *ara* C^{r-1} *map*

Proof. The results follow by an application of the method of liapunou-schmidt, (McGhee and Sallam 1988) to find solution of the form: $x = ux^{\circ} + x^1$, where $x^1 \in x_1$.

For the sake of completeness we will proceed with the proof, suitably modified fo problem let Q_o and Q_1 be the projections of y onto y_o and y_1 respectively (see (3.4)). T

$$\begin{aligned} & M(\underline{\lambda}; x) = 0 \\ \Leftrightarrow\ & Q_1 M(\underline{\lambda}; x) = 0 \\ \text{and}\ & Q_o M(\underline{\lambda}; x) = 0 \end{aligned}$$

the so called auxiliary equation and bifurcation equation, respectively. The auxiliary equ becomes

$$Q_1 w(\underline{\lambda}) x_1 - Q_1 f((\underline{\lambda} m, \underline{\mu} m), ux^{\circ} + x^1) = 0,$$

where $u \in \mathbb{R}, x^1 \in X_1$.

Consider the mapping $\psi : \mathbb{R} \times \mathbb{R}^{n-m} \times \mathbb{R} \times X_1 \longrightarrow y_1$ defined by

$$\psi(\underline{\lambda} m, \underline{\mu} m u, x^1) = Q_1 w(\underline{\lambda}) x_1 - Q_1 f((\underline{\lambda} m, \underline{\mu} m), ux^{\circ} + x^1) = 0.$$

Using $C2$ and $C3$ we obtain

$$\psi(\underline{\lambda} m, \underline{\mu} \overset{\circ}{m}, 0, 0) = 0$$
$$D_{x^1} \psi(\underline{\lambda} \overset{\circ}{m}, \underline{\mu} \overset{\circ}{m}, 0, 0) = Q_1 W(\underline{\lambda}^{\circ}).$$

Since $Q_1 W(\underline{\lambda}^{\circ}) : X_1 \longrightarrow y_1$ is a bounded linear isomorphism, it follows fro implicit function theorem that there exists a neighborhood $U C \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}$ of $, \underline{\mu} m^{\circ}, 0)$ and a unique mapping $x^{1*} \in C(U, X_1)$ such that

$$x^{1*}(\underline{\lambda} \overset{\circ}{m}, \underline{\mu} \overset{\circ}{m}, 0) = 0$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

and
$$\psi(\underline{\lambda}m, \underline{\mu}m, ux^{1*}(\underline{\lambda}m, \underline{\mu}m, u)) = 0,$$
i.e., the auxiliary equation (3.8) is satisfied since by $C_2, (\underline{\lambda}\, m^\circ, \underline{\mu}\, m^\circ, 0, 0)$ satisfies (3.7) and by the implicit function theorem x^{1*} is unique, it follows that
$$x^{1*}(\underline{\lambda}m, \underline{\mu}m, 0) = 0 \forall (\underline{\lambda}m, \underline{\mu}m, 0) \in U. \tag{3.10}$$

Differentiation of (3.9) with respect to u and using $C3$ and (3.10) gives
$$Q_1 W(\underline{\lambda}m, \underline{\mu}\, \overset{\circ}{m}) D_u x^{1*}(\underline{\lambda}\, \overset{\circ}{m}, \underline{\mu}\, \overset{\circ}{m}, 0) = 0.$$

Since for $\|\underline{\lambda} - \underline{\lambda}^\circ\|$ sufficiently small $Q_1 W(\underline{\lambda}m, \underline{\mu}\, m^\circ)$ is a bounded linear isomorphism of X_1 onto y_1 we can conclude that
$$D_u x^{1*}(\underline{\lambda}m, \underline{\mu}\, \overset{\circ}{m}, 0) = 0 \text{ for } \|\underline{\lambda}m - \underline{\lambda}\, \overset{\circ}{m}\| \quad \text{sufficiently small.} \tag{3.11}$$

Differentiating (3.10) repeatedly with respect to $\underline{\mu}m$ gives
$$D^k_{\mu m} x^{1*}(\underline{\lambda}m, \underline{\mu}m, 0), 1 \le k \le r. \tag{3.12}$$

Using (3.10)-(3.12) we see, from Taylor's theorem $x^{1*}(\underline{\lambda}m, \underline{\mu}m, u) = 0(|u|^2 + |u| \|\underline{\mu}m - \underline{\mu}\, m^\circ\|)$ as $|u|, \|\underline{\mu}m - \underline{\mu}\, m^\circ\| \longrightarrow 0$, where $x^{1*}(\underline{\lambda}m, \underline{\mu}m, u) = (x_1^{1*}(\underline{\lambda}m, \underline{\mu}m, u), \ldots, x^{1*}m(\underline{\lambda}m, \underline{\mu}m, u)$.

The bifurcation equation becomes
$$\begin{aligned} & Q_\circ W(\underline{\lambda})(ux^\circ + x^{1*}(\underline{\lambda}m, \underline{\mu}m, u)) - Q_\circ F(\underline{\lambda}m, \underline{\mu}m, ux^\circ + x^{1*}(\underline{\lambda}m, \underline{\mu}m, u)) = 0 \\ \Leftrightarrow \quad & -u\sum_{j=1}^{m}(\underline{\lambda}_j - \underline{\lambda}_j^\circ) B_j x^\circ - u \sum_{j=m+1}^{n} (\underline{\lambda}_j - \underline{\lambda}_j^\circ) B_j x^\circ + Q_\circ N(\underline{\lambda}) x^{1*}(\underline{\lambda}m, \underline{\mu}m, u) \\ & - Q_\circ F((\underline{\lambda}m, \underline{\mu}m), ux + x(\underline{\lambda}\, \overset{\circ}{m}, \underline{\mu}m^{1*}, u)) = 0. \end{aligned} \tag{3.13}$$

Using the basis vectors $B_j x^\circ, i = 1, \ldots, m$, the bifurcation functions $F = (F, \ldots, Fm) : \mathbb{R}^m * \mathbb{R}^{n-m} * \mathbb{R} \longrightarrow \mathbb{R}^m$ is defined by
$$\sum_{j=1}^{m} F_j(\underline{\lambda}m, \underline{\mu}m, u) B_j x^\circ := -u \sum_{j=1}^{m} (\underline{\lambda}_j - \underline{\lambda}_j^\circ) B_j x^\circ - \sum_{j=1}^{m} G_j(\underline{\lambda}m, \underline{\mu}m, u),$$
where
$$\sum_{j=1}^{m} G_j(\underline{\lambda}m, \underline{\mu}m, u) B_j x^\circ := Q_\circ \Big\{ u \sum_{j=m+1}^{n} (\underline{\lambda}_j^\circ - \underline{\lambda}_j) B_j x^\circ - W(\underline{\lambda}) x^{1*}(\underline{\lambda}m, \underline{\mu}m, u) + f((\underline{\lambda}m, \underline{\mu}m), ux^\circ + x^{1*}(\underline{\lambda}m, \underline{\mu}m, u) \Big\}.$$

Thus, $F(\underline{\lambda}m, \underline{\mu}m, u) = -u(\underline{\lambda}m - \underline{\lambda}\, m^\circ) - G(\underline{\lambda}m, \underline{\mu}m, u)$ where $G = (G_1, \ldots, Gm) : \mathbb{R}^m * \mathbb{R}^{n-m} * \mathbb{R} \longrightarrow \mathbb{R}^m$ satisfies $G(\underline{\lambda}m, \underline{\mu}m, 0) = 0$, so that $D^k_{\underline{\mu}m} G(\underline{\lambda}m, \underline{\mu}m, 0) = 01 \le k \le r$ and $D_u G(\underline{\lambda}\, m^\circ, \underline{\mu}\, m^\circ, 0) = 0$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

It follows that $G(\underline{\lambda}m, \underline{\mu}m, u) = u\widetilde{G}(\underline{\lambda}m\underline{\mu}m, u)$ where $\widetilde{G} \leftarrow C^{r-1}(\mathbb{R}^m * \mathbb{R}^{n-m} * \mathbb{R}$, $\mathbb{R}^m)$, and the bifurcation equations reduces to $H(\underline{\lambda}m, \underline{\mu}m, u) := -(\underline{\lambda}m - \underline{\lambda}\, m^\circ) - \widetilde{G}(\underline{\lambda}m, \underline{\mu}m, 0)$ $= 0$.

Now, $\widetilde{G}(\underline{\lambda}\, m^\circ, \underline{\mu}\, m^\circ, 0) = 0$ and $D_{\underline{\lambda}m}\widetilde{G}(\underline{\lambda}m, \underline{\mu}m, 0) = 0$.

So that $H(\underline{\lambda}\, m^\circ, \underline{\mu}\, m^\circ, 0) = 0$ and $D_{\underline{\lambda}m}H(\underline{\lambda}\, m^\circ, \underline{\mu}\, m^\circ, 0) = -Idm$, where Idm denotes the identity mapping on $\mathbb{R}^m$.

Therefore, by the implicit function theorem, there is a neighborhood $VC\, \mathbb{R}^{n-m} * \mathbb{R}$ of $(\underline{\mu}\, m^\circ, 0)$ and a unique function $\underline{\lambda}^* m \leftarrow C^{r-1}(V, \mathbb{R}^m)$ such that $H(\underline{\lambda}^* m(\underline{\mu}m, u) = 0 \forall (\underline{\mu}m, u) \in V$ and $\underline{\lambda}^* m(\underline{\mu}m, u) = \underline{\mu}\, m^\circ + 0(|u| + \|\underline{\mu}m - \underline{\mu}\, m^\circ\|)$.

The (3.5) has non-trivial solutions $(\underline{\lambda}^* m(\underline{\mu}m, u), \underline{\mu}m), x^{1*}(\underline{\mu}m, u) \in \mathbb{R}^n * x : (\underline{\mu}m, u) \in V$, where $\underline{\lambda}^* m(\underline{\mu}m, u) = \underline{\lambda}\, m^\circ + 0(|u| + \|\underline{\mu}m - \underline{\mu}\, m^\circ\|)$ and $x^{1*}(\underline{\mu}m, u) = ux^\circ + 0(|u|^2 + |u|\|\underline{\mu}m - \underline{\mu}\, m^\circ\|)$ as $\|\underline{\mu}m - \underline{\mu}\, m^\circ\|, |u| \longrightarrow 0$.

We also have the following result on linear perturbation of the system (1.1) and the proof is again similar to that of Theorem 3.2.

THEOREM 3.3. *Suppose $H_i, i = 1, \ldots, k$ are Hilbert spaces $A_i(\epsilon), B_{ij}(\epsilon) : H_i \longrightarrow H_i$ are bounded self-adjoint linear operators continues in $\underline{\epsilon}$ (or C^r resp analytic), $\underline{\epsilon} \in \mathbb{G}^l$, for $\underline{\epsilon}$ in an open neighborhood of $0 \in \mathbb{G}^l$. If $\underline{\lambda} = \underline{\lambda}^\circ \in \mathbb{R}^\circ$ is such that each of the problems in system (3.3) has a solution $x_i^\circ \neq 0$ and the conditions of lemma (3.1) are satisfied, then there is a constant $\delta > 0$ such that for $\underline{\epsilon} \in \mathbb{G}^l, \|\underline{\epsilon}\| < \delta, \|\underline{\mu}m - \mu^\circ\| < \delta$, there is a bifurcation i.e.*

Solutions $(\underline{\lambda}\, m^\circ\, (\epsilon, \underline{\mu}m), \underline{\mu}mx^*(\underline{\epsilon}, \underline{\mu}m))$ of the system

$$(A_i(\epsilon) - \sum_{j=1}^{n} \lambda_j A_{ij}(\epsilon))x_i = 0 \quad i = 1, \ldots, m$$

with $(\lambda\, m^*\, (0, \underline{\mu}^\circ), \underline{\mu}\, m^\circ) = (\underline{\lambda}\, m^\circ, \underline{\mu}\, m^\circ) = \underline{\lambda}^\circ, x^*(0, \underline{\mu}\, m^\circ) = x^\circ$.

The functions $\underline{\lambda}^* m(\underline{\epsilon}, \underline{\mu}m), x^*(\epsilon, \underline{\mu}m)$ are continuous in $\underline{\epsilon}$ (or c^r resp analytic).

References

Binding, P. (1984) : Perturbation and bifurcation of non-singular multiparamevric eigenvalues; *Non linear Anal. 8*, 335.

Browne, P. J. (1977) : A completeness Theorem for a Non-linear Multiparameter eigenvalue problem. *J. Diff. Eqns. 23*, 28[illegible]

Browne P. J. and **Sleeman, B. D.** (1979) : Non linear multiparameter Sturm–Liouville Problems. *J. Diff. Eqns. 34*, 139.

Browne, P. J. (1980) : Non-linear multiparameter eigenvalue problems for ordinary differntial equations. *J. Math. Anal. and Applic. 77*, 425.

Browne, P. J. and **Sleeman, B. D.** (1980) : Bifurcation from eigenvalues in non-linear multiparameter sturm–Liouville Problems; *Glasgow Math. J. 21*, 85.

Browne, P. J. and **Sleeman, B. D.** (1981) : Non-linear multiparameter eigenvalue problems. *Quaestiones Math. 4*, 277.

McGhee, D. F. and **Sallam, M. H.** (1988) : Simple eigenvalues and bifurcation for a multiparameter problem. *Proceedings of the Edinburgh Math. Soc. 31*, 77.

FACULTY OF SCIENCE
DEPARTMENT OF ELECTRONICS AND COMMUNICATION
MENOUFIA UNIVERSITY
MENOUFIA

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc., 86, 233–236 (1994)

AN EXTENSION OF A BILATERAL GENERATING FUNCTION OF MODIFIED JACOBI POLYNOMIAL

RATNA CHAKRAVARTY (HAZRA) AND D. K. BASU

(*Received 25 July 1992*)

Abstract. A new class of generating functions involving modified Jacobi and Laguerre polynomials have been obtained.

Introduction. While extending the general theorem on bilateral generating function for the modified Jacobi polynomials $P_n^{(\alpha-n,\beta)}(x)$, we introduce the bilateral generating function for the modified Jacobi and Laguerre polynomials by means of the relation

$$G(x,z,w)=\sum_{n=0}^{\infty} a_n w^n P_n^{(\alpha-n,\beta)}(x) L_m^{(n)}(z)$$

and we prove that the existence of such a bilateral generating function implies the existence of a more general generating function.

In 1988, B.Ghosh obtained the following bilateral generating function by classical method:

THEOREM 1. *If* $F(x,t)=\sum_{n=0}^{\infty} a_n P_n^{(\alpha-n,\beta)}(x)t^n$ *then*

$$\sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta)}(x)\cdot\sigma_n(y)t^n=(1+t)^{\alpha}\left(1+\frac{t}{2}(1-x)\right)^{-1-\alpha-\beta} F\left(\frac{x-\frac{t}{2}(1-x)}{1+\frac{t}{2}(1-x)},\frac{yt}{1+t}\right)$$

where $\sigma_n(y)=\sum_{p-0}^{n} a_p\binom{n}{p}y^p$.

Research party supported by U.G.C., New Delhi, under U.F. No. 26-1(41)/91.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The importance of the theorem lies in the fact that all particular bilateral (or biline generating functions can be easily obtained by giving suitable values to a_n and then by maki use of known linear generating relations involving modified Jacobi polynomials.

The object of the present paper is to extend the above theorem further in the followi form:

THEOREM 2. *If there exists the following bilateral generating function for the modifi Jacobi and Laguerre polynomials* $G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n p_n^{(\alpha-n,\beta)}(x) L_m^{(n)}(z)$ *then the followi more general generating function will hold:*

$$(1-2w)^{\alpha}\left[1-w(1-x)\right]^{-\alpha-\beta-1} e^{-w} G\left(\frac{x+w(1-x)}{1-w(1-x)}, z+w, \frac{wv}{1-2w}\right)$$

$$= \sum_{n,p,q=0}^{\infty} \frac{(-1)^{p+q} 2^p (n+1)_p}{p!\, q!} a_n\, w^{p+q+n} v^n P_{n+p}^{(\alpha-n-p,\beta)}(x) L_m^{(n+q)}(z).$$

2. Proof of Theorem 2. Let $G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n P_n^{(\alpha-n,\beta)}(x) L_m^{(n)}(z)$.

Replacing w by $wytv$ we have

$$G(x,z,wytv) = \sum_{n=0}^{\infty} a_n (wv)^n P_n^{(\alpha-n,\beta)}(x) y^n L_m^{(n)}(z) t^n. \tag{2.}$$

Now, we choose the following two operators (Ghosh 1986, Mcbride 1971):

$$R_1 = (1-x^2) y \frac{\partial}{\partial x} + 2y^2 \frac{2\partial}{\partial y} + (1-x)(\alpha+\beta+1)y - 2\alpha y \quad \text{and} \quad R_2 = t\frac{\partial}{\partial z} - t$$

so that $R_1\left[P_n^{(\alpha-n,\beta)}(x) y^n\right] = -2(n+1) P_{n+1}^{(\alpha-n-1,\beta)}(x) y^{n+1}$ and $R_2\left[L_m^{(n)}(z) t^n\right] = -L_m^{(n+1)}(z) t^{n+}$

The extended form of the transformation group generated by R_1 and R_2 are, respe tively,

$$e^{wR_1} f(x,y) = (1-2wy)^{\alpha}\left[1-wy(1-x)\right]^{-\alpha-\beta-1} f\left(\frac{x+wy(1-x)}{1-wy(1-x)}, \frac{y}{1-2wy}\right)$$

and $\quad e^{wR_2} f(z,t) = e^{-wt} f(z+wt, t).$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Operating both sides of (2.1) by $e^{wR_1}e^{wR_2}$

The left member of (2.1) becomes

$$(1-2wy)^{\alpha}\left[1-wy(1-x)\right]^{-\alpha-\beta-1}e^{-wt}G\left(\frac{x+wy(1-x)}{1-wy(1-x)}, z+wt, \frac{wytv}{1-2wy}\right).$$

On the other hand, right side of (2.1) reduces to

$$\sum_{n,p,q=0}^{\infty}\frac{(-1)^{p+q}2^p(n+1)_p}{p!\,q!}a_n w^{p+q+n}v^n;$$

$$P_{n+p}^{(\alpha-n-p,\beta)}(x)y^{n+p}L_m^{(n+q)}(z)t^{n+q}.$$

Equating and putting $y=t=1$ we obtain Theorem 2.

3. Application. If we put $m=0$, we notice from our theorem that $G(x,z,w)$ becomes $G(x,w)$ for $L_0^{(n)}(z)=1$. Hence from theorem 2 we obtain:

$$(1-2w)^{\alpha}\left[1-w(1-x)\right]^{-\alpha-\beta-1}G\left(\frac{x+w(1-x)}{1-w(1-x)}, \frac{wv}{1-2w}\right)$$

$$=\sum_{n,p=0}^{\infty}\frac{w^{p+n}}{p!}v^n a_n(-1)^p 2^p(n+1)_p P_{n+p}^{(\alpha-n-p,\beta)}(x)$$

Thus, we have

$$(1+t)^{\alpha}\left[1+\frac{t}{2}(1-x)\right]^{-\alpha-\beta-1}G\left(\frac{x-\frac{t}{2}(1-x)}{1+\frac{t}{2}(1-x)}, \frac{ty}{1+t}\right)=\sum_{n=0}^{\infty}P_n^{(\alpha-n,\beta)}(x)\sigma_n(y)t^n$$

where $\sigma_n(y)=\sum_{m=0}^{n}\binom{n}{m}a_m y^m$ which was derived by B. Ghosh.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

References

Ghosh, B. (1988) : *Bull. Cal. Math. Soc.* 80, 3, 215.

Ghosh, B. (1986) : *Bull Cal. Math. Soc.* 78, 3, 187.

Mcbride, E. B. (1971) : *Obtaining Generating Functions,* Springer Verlag.

SANTRAGACHI KEDARNATH INSTITUTION (GIRLS')
181, SASTRI NARENDRA GANGULY ROAD
HOWRAH 711104

AND

DEPARTMENT OF MATHEMATICS
BANGABASI EVENING COLLEGE
19 RAJKUMAR CHAKRABARTY SARANI
CALCUTTA 700009

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc., 86, 237–242 (1994)

EXISTENCE THEOREMS, UNIQUENESS FOR GENERAL FORMULATION OF THE MINIMUM EFFORT PROBLEMS IN LOCALLY CONVEX LINEAR TOPOLOGICAL SPACES

GARGI CHAKRABORTY

(*Received 25 November 1991; revision received 19 October 1992*)

Abstract. In this paper, a class of generalised minimum effort problems has been formulated in locally convex linear topological spaces. Some necessary conditions has been obtained. Existence and uniqueness of the solutions are discussed.

Introduction. Neustadt (1992), Porter and Williams (1966), Minamide and Nakamura (1972), Burns (1975) considered generalised minimum effort control problems in Banach space setting. The author has obtained some results of a linear control problem in locally convex linear topological space (Chakraborty 1989). The principle objective of this paper is to generalise the idea of minimum effort problem in locally convex linear topological space setting. Some necessary conditions regarding the existence and uniqueness of the solution has been obtained. Fundamental tools of the functional analysis is used to solve the problem.

Here we state the problem:

PROBLEM 1. Let L and L' be two locally convex linear topological space and T, a continuous linear transformation from L into L'. For each ξ in the range of T, we find an element $l \in L$ that satisfies $\xi = Tl$ while minimizing $J(l) = \sup\left\{| < l, f > |, f \in E\right\}$ where E is any subset of L^*, L^* being the conjugate space of L. To solve the above problem, we assume the following:

(i) E_π is the polar set in L of any arbitrary bounded subset E of L^*,

(ii) for every neighbourhood, say, U of zero in $L, T(U)$ contains some set that is of the second category in L' and that satisfies the condition of Baire

NOTE 1. From (i), it is clear that E_π is a neighbourhood of zero (Yosida 1965, p. 139) and so E_π is bounded (Kelley et al 1963, p. 44).

In (Chakraborty 1989), it has been shown that there exists an element $l_\xi \in L$ for each $\xi \in L'$ satisfying $\xi = Tl_\xi$ and which minimizes $J(l) = \sup\left\{| < l, f > |, f \in E_\pi^\pi\right\}$, $E_\pi^\pi \subset L^*$ provided L is semi-reflexive $\left[E_\pi^\pi \text{ denotes polar set of } E_\pi \text{ (Yosida 1965)}\right]$.

NOTE 2. In (Chakraborty 1989) it has been shown that L is a Barrelled space. As L is also semi-reflexive, consequently L is reflexive (Yosida 1965, p. 140).

Since the character of L is determined by E_π, let us introduce some definitions:

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Definition of rotund space. A locally linear topological space L is rotund if the boundary of E_π, i.e., $\partial E_\pi = \left\{ l \in L : \sup\{| < l, f > |, f \in E\} = 1 \right\}$ contains no line segments.

It is easy to verify that the above definition of Rotund is equivalent to the following statement.

Each hyperplane of support of E_π meets E_π in at most one (and exactly one) point. Now it will be shown that l_ξ is unique in the following theorem.

THEOREM A. *Let L and L' be two locally convex linear topological spaces and let $T : L \to L'$ be a continuous linear transformation. Also, let L be semi-reflexive and rotund. Then for each $\xi \in L'$ there exists a unique element $l_\xi \in \partial E_\pi$ such that $l_\xi = T^\dagger(\xi)$ while minimizing $J(l) = \sup\left\{| < l, f > |, f \in E_\pi^\pi\right\}$.* [We write $T^\dagger(\xi)$ for the unique pre-image of a vector ξ in the range of T.]

Proof. In (Chakraborty 1989) we have proved that there exists an element $l_\xi \in \partial E_\pi$ for each $\xi \in L'$ such that $Tl_\xi = \xi$ minimizing $J(l) = \sup\left\{| < l, f > |, f \in E_\pi^\pi\right\}$.

Now we have to show that this element l_ξ is unique. Let l_1 and l_2 be two elements in $\partial E_\pi \subset L = L^{**}$ [Since L is reflexive] satisfying $Tl = \xi$ and $J(l) = \sup\left\{| < l, f > |, f \in E_\pi^\pi\right\}$.

Since E_π is convex (Chakraborty 1989), then $l = \alpha l_1 + (1-\alpha) l_2, \quad 0 < \alpha < 1$.

By definition of minimization, $J(l) \geq J(l_1) = J(l_2)$.

Now,

$$\begin{aligned} | < l, f > | &= | < \alpha l_1 + (1-\alpha) l_2, f > | = | < \alpha l_1, f > + < (1-\alpha) l_2, f > | \\ &\leq | < \alpha l_1, f > | + | < (1-\alpha) l_2, f > | \\ &= \alpha | < l_1, f > | + (1-\alpha) | < l_2, f > |. \end{aligned}$$

$$\begin{aligned} \text{So,} \quad \sup\left\{| < l, f > |, f \in E_\pi^\pi\right\} &\leq \alpha \sup\left\{| < l_1, f > |, f \in E_\pi^\pi\right\} \\ &\quad + (1-\alpha) \sup\left\{| < l_2, f > |, f \in E_\pi^\pi\right\}. \end{aligned}$$

$$\therefore \quad J(l) = \alpha J(l_1) + (1-\alpha) J(l_2).$$

From the convexity relationships and the triangle inequality, we have

$$J(l) = J(l_1) = J(l_2).$$

Since L is rotund then the polar set ∂E_π contains no line segments and hence $l = l_1 = l_2$. Thus, l is a unique element in L for each $\xi \in L'$ and hence $l = T^\dagger \xi$.

Therefore there exists a unique elements $l_\xi \in L$ for each $\xi \in L'$ such that $l_\xi = T^\dagger \xi$ while minimizing $J(l) = \sup\left\{| < l, f > |, f \in E_\pi^\pi\right\}$.

Throughout this paper we use the following: L, L_1, L_2 and L' are all locally convex linear topological spaces. E_π is the polar set of bounded set $E \subset L^*$ in L where L^* is the conjugate space of L.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In this paper we further assume the hypothesis (i), (ii) [see p. 237].

PROBLEM 1(a). Let L, L', T and ξ be as before. Also let L be semi-reflexive and rotund and $\hat{l}$ be a given vector in L. Then the problem is to find a $l \in L$ satisfying $Tl = \xi$ and which minimizes $J(l) = \sup\left\{| < l - \hat{l}, f > |, f \in E_\pi^\pi\right\}$.

Now, let us consider uniqueness of the above problem.

THEOREM B. *The problem* 1(*a*) *has a unique solution of the following form:*

$$l_\xi = T^\dagger(\xi - T\hat{l}) + \hat{l}.$$

Proof. Evidently, l_ξ maps into ξ under T. Now let l be any pre-image of ξ under T. Then $T(l - \hat{l}) = \xi - T\hat{l}$.

Now, by definition of $T^\dagger$, the above equality implies that

$$\sup\left\{| < l - \hat{l}, f > |, f \in E_\pi^\pi\right\} \geq \sup\left\{| < T^\dagger(\xi - T\hat{l}), f > | f \in E_\pi^\pi\right\}$$
$$= \sup\left\{| < (l_\xi - \hat{l}), f > |, f \in E_\pi^\pi\right\}$$

where $l_\xi = T^\dagger\xi$ [p. 238].

Then, it follows that l_ξ is a solution of the above problem.

Since $l_\xi - \hat{l} = T^\dagger(\xi - T\hat{l})$, i.e., $l_\xi = T^\dagger(\xi - T\hat{l}) + \hat{l}$ the above inequality is not strict and hence l_ξ is the unique solution such that $Tl_\xi = \xi$ while minimizing

$$J(l) = \sup\left\{| < l - \hat{l}, f > |, f \in E_\pi^\pi\right\}.$$

NOTE 3. Product of two locally convex linear topological spaces L and L_1 is denoted by $L \times L_1$ which is also a locally convex linear topological space (Kelly et al 1963). Addition and scalar multiplication in $L \times L_1$ are in the usual way:

$$(l_1, v_1) + (l_2, v_2) = (l_1 + l_2, v_1 + v_2)\alpha(l, v) = (\alpha l, \alpha v).$$

LEMMA A. (a) *Let $F : L \to L_1$ be a continuous linear transformation. Then the graph of F, $L(F) = \left\{(l, v) : v = Fl, l \in L\right\}$ is a closed subspace of $L \times L_1$.*

(b) *When L is semi-reflexive and rotund then $L(F)$ is semi-reflexive and rotund.*

(c) *Let $T : L \to L'$ be a continuous linear transformation. If $G : L(F) \to L'$ is a transformation defined by $G\left[(l, Fl)\right] = Tl, l \in L$, then G is a continuous linear transformation from $L(F)$ into L'.*

Proof of Lemma A(a). The graph of F, $L(F) = \left\{(l, v) : v = Fl, l \in L\right\}$ is a linear subspace of $L \times L_1$ and hence $L(F)$ is locally convex linear topological space (Kelley et al 1963 p. 47). Now, we shall show that $L(F)$ is closed in $L \times L_1$.

Let $(l_n, Fl_n) \to (l, v)$.

Then $l_n \to l, Fl_n \to v$.

Since F is continuous, then $l_n \to l \Rightarrow Fl_n \to Fl = v$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Hence $(l, Fl) \in L(F)$ which implies that $L(F)$ is closed.

Proof of Lemma A(b). Since L is semi-reflexive and E_π is weakly closed, bounde hence E_π is weakly compact in L (Chakraborty 1989), (Kelley et al 1963). Again, F is nd continuous linear transformation and hence F is a weakly continuous linear transformatio Therefore $F(E_\pi)$ is weakly compact set in L_1. Consequently, $E_\pi \times F(E_\pi)$ is weakly compa $(l, F$ set in $L(F)$ and hence $L(F)$ is semi-reflexive (Kelley et al 1963).

Here we have to show that $L(F)$ is rotund. If possible, let $L(F)$ be not rotund. The there must exist at least one supporting hyperplane say, $F = (f_1, f_2)$ which will meet $E_\pi \times F(E$ [a convex body in $L(F)$] in two points, say (l_1, Fl_1) and (l_2, Fl_2) where $l_1 \neq l_2$.

But this is impossible as L is rotund [p. 238]. Hence $L(F)$ is rotund.

Proof of Lemma A(c).

$$G\Big[(l_1 + l_2), F(l_1 + l_2)\Big] = T(l_1 + l_2), l_1, l_2 \in E = Tl_1 + Tl_2$$
$$= G\Big[(l_1, Fl_1)\Big] + G\Big[(l_2, Fl_2)\Big].$$

Again, $G\Big[\alpha(l, Fl)\Big] = G\Big[(\alpha l, \alpha Fl)\Big] = T(\alpha l) = \alpha Tl = \alpha G\Big[(l, Fl)\Big]$.

Therefore, G is a linear transformation.

Also, if $(l_n, Fl_n) \to (l, Fl)$ we shall show that $G\Big[(l_n, Fl_n)\Big] \to G\Big[(l, Fl)\Big]$.

Now, $G\Big[(l_n, Fl_n)\Big] = Tl_n, l_n \in L$.

Since T is a continuous linear transformation, then $l_n \to l \Rightarrow Tl_n \to Tl$ and henc $G\Big[(l_n, Fl_n)\Big] \to Tl = G\Big[(l, Fl)\Big]$.

Therefore, G is continuous linear transformation from $L(F)$ into L'.

PROBLEM 2. Let L be semi-reflexive and rotund.

Let $F : L \to L_1$ and $T : L \to L'$ be two continuous linear transformations. Let $\xi \in L$ be any vector. Then the problem is to find the element $l \in T^{-1}(\xi) \subset L$ of all pre-images of such under T while minimizing

$$J(l) = \sup\Big\{| < l, f > |, f \in E_\pi^\pi \subset L^*\Big\} + \sup\Big\{| < Fl, g > |, g \in \big(F(E_\pi)\big)^\pi \subset L_1^*\Big\}.$$

With the help of Lemma A, the above problem can be formulated in the following way

Let $G : L(F) \to L'$ be a continuous linear transformation. Then the problem is to fin a vector $(l, Fl) \in L(F)$ for any vector $\xi \in L'$ such that $G\Big[(l, Fl)\Big] = \xi$ and which minimizes

$$J(l) = \sup\Big\{| < (l, Fl), (f, g) > |, (f, g) \in E_\pi^\pi \times \big(F(E_\pi)\big)^\pi\Big\}. \quad (1$$

Thus, the problem 2 can be reduced to problem 1, where T is replaced by G $G : L(F) \to L'$. beca

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Evidently, hypothesis (i) and (ii) of Problem 1 are satisfied for $\left(E_\pi^\pi \times (F(E_\pi))^\pi\right)_\pi$ and G.

Hence the Problem 2 has a unique solution, that is, there exists a unique pre-image $(l, Fl) \in L(F)$ for each $\xi \in L'$, such that $G\left[(l, Fl)\right] = \xi$ and which minimizes

$$J(l) = \sup\left\{|<(l,Fl),(f,g)>|, (f,g) \in E_\pi^\pi \times (F(E_\pi))^\pi\right\}$$
$$= \sup\left\{|<l,f>|, f \in E_\pi^\pi \subset L^*\right\} + \sup\left\{|<Fl,g>|, g \in (F(E_\pi))^\pi \subset L_1^*\right\}.$$

NOTE 4. In general,

$$\left(E_\pi^\pi \times (F(E_\pi))^\pi\right)_\pi \subset E_{\pi\pi}^\pi \times \left(F(E_\pi)\right)_\pi^\pi.$$

I have got this relation from the suggestions of reviewer, (Analysis, International Journal of Analysis and its Applications) on the paper 'An optimal control problem in locally convex linear topological spaces'. Manuscript No. KR48/90, June 7, 1990].

Also, $E_{\pi\pi}^\pi = E_\pi$, $(F(E_\pi))_\pi^\pi = F(E_\pi)$ [(Kelley et al 1963), p. 141]

$$\therefore \quad E_{\pi\pi}^\pi \times (F(E_\pi))_\pi^\pi = E_\pi \times F(E_\pi).$$

But since $L(F)$ is semi-reflexive and rotund,

$$\therefore \quad \left(E_\pi^\pi \times (F(E_\pi))\right)_\pi^\pi = E_\pi \times F(E)_\pi.$$

Again, let

$$(l, Fl) \in E_\pi \times F(E_\pi) = \left(E_\pi^\pi \times (F(E_\pi))^\pi\right)_\pi$$

such that

$$\sup\left\{|<(l,Fl),(f,g)>|, (f,g) \in E_\pi^\pi \times (F(E_\pi))^\pi\right\} \leq 1.$$

Then,

$$(l, Fl) \in \left(E_\pi^\pi \times (F(E_\pi))^\pi\right)_\pi \quad \text{and}$$
$$\sup\left\{|<(l,Fl),(f,g) \in E_\pi^\pi \times (F(E_\pi))^\pi\right\}$$
$$= \sup\left[\left\{|<l,f>|, f \in E_\pi^\pi\right\} + \left\{|<Fl,g>|, g \in (F(E_\pi))^\pi\right\}\right]$$
$$= \sup\left\{|<l,f>|, f \in E_\pi^\pi\right\} + \sup\left\{|<Fl,g>|, g \in (F(E_\pi))^\pi\right\}$$

because $E_\pi^\pi \times (F(E_\pi))^\pi, E_\pi^\pi$ and $(F(E_\pi))^\pi$ are bounded.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

THEOREM C. *Problem 2 has a unique solution $l_\xi \in L$, namely, the argument of vector $G^\dagger(\xi)$ in $L(F)$ for each $\xi \in L'$.*

Next, we shall consider the following problem analogous to Problem 1(a).

PROBLEM 2(a). Let L, be semi-reflexive and rotund.

Let $F : L \to L_1$ and $T : L \to L'$ be two continuous linear transformations. Let $\xi \in$ and $(\hat{l}, \hat{y}) \in L \times L_1$ be given. Then the problem is to find a $l \in L$ such that $Tl = \xi$ whi minimizes

$$J(l) = \sup\left\{| < l - \hat{l}, f > |, f \in E_\pi^\pi \subset L^*\right\} + \sup\left\{| < Fl - \hat{y}, g > |, g \in \left(F(E_\pi)\right)^\pi \subset L_1^*\right\}.$$

To solve the Problem 2(a) we consider the graph $L(F) \subset L \times L_1$ and the transformati $G : L(F) \to L'$.

Put $\hat{w} \in L(F)$. Then the Problem 2(a) may now be restated as follows:

Find $w = (l, Fl) \in L(F)$ satisfying $G(w) = \xi$ and which minimizes

$$\sup\left\{| < (w - \hat{w}), h > |, h = (f, g) \in \left(E_\pi^\pi \times \left(F(E_\pi)\right)^\pi\right) \subset (L \times L_1)^*\right\}.$$

Now, if $\hat{w} \in L(F)$, then the Problem 2(a) can be considered as Problem 1(a) wh L, T, l and $\hat{l}$ are replaced by $L(F), G, w$ and $\hat{w}$, respectively. Hence the unique solution of t Problem 2(a) is $(l_\xi, Fl_\xi) = G^\dagger(\xi - G\hat{w}) + (\hat{l}, F\hat{l})$.

Acknowledgement. The author is most grateful to Dr. R. N. Mukherjee of t Department of Mathematics, University of Burdwan (W.B.), for his generous encourageme and suggestions at various steps while preparing this paper.

References

Burns, J. A. (1975) : *Journal of Optimization Theory and Application, 15(4),* 413.

Chaudhuri, A. K. and **Mukherjee, R. N.** (1981) : *Indian J. pure appl. Math., 12,* 151.

Chakraborty, G. (1989) : *Indian J. pure appl. Math., 20(8),* 761.

Day, M. M. (1962) : *Normal Linear Spaces,* Springer-Verlag, Berlin.

Kelley, J. L. and **Namioka Issac** and **Co-authors** (1963) : *Linear topological Space,* D. Van Nostand Co. Inc.

Minamide, N. and **Nakamura, K.** (1972) : Linear Bounded Phase Co-ordinate Control Problems under Certain Regular and Normality Conditions, *SIAM, Journal of Control, 10,* 82.

Neustadt, L. W. (1962) : Minimum Effort Control Systems, *J. Soc. Ind. Appl. Math. Control, Ser. Al(1),* 16.

Porter, W. A. and **Williams, J. P.** (1966) : A Note on the Minimum Effort Control Problem, *J. Math. Anal. Appl., 13,* 25

Yosida, Kosaku (1965) : *Functional Analysis,* Springer-Verlag, Berlin.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BURDWAN
BURDWAN 713104 (W.B.)
INDIA

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

ll. Cal. Math. Soc., 86, 243–248 (1994)

RAYLEIGH WAVES IN A MAGNETOELASTIC INITIALLY RESSED CONDUCTING MEDIUM WITH THE GRAVITY FIELD

A.M.EL-NAGGAR, A.M.ABD-ALLA AND S. M. AHMED

(*Received 10 February1993; revision received 3 September 1993*)

1. Abstract. In this paper an equation is derived determining the velocity of propa- ion of sinusoidal Rayleigh waves including the initial stress, gravity field and magnetic field, a homogeneous isotropic elastic half-space, subject to certain boundary condition. The re- t obtained is in agreement with the corresponding result, when the gravity field is neglected, reover when the initial stress and the magnetic field are neglected.

1. Introduction. The problem of elastic wave propagation in a magnetic field was estigated by Chadwick (1957). Also, it has been shown that the initial stress has a pro- nced effect on the elastic wave propagation (Biot 1940). Yu, Tang (1966) considered the blem of elastic wave propagation in an initially stressed conducting solid, to which a static gnetic field is applied. The interaction between the elastic wave motions and a magnetic d in electrical conductors was investigated by Knopoff (1955). The problem of determining existence of the magnetoelastic Rayleigh waves in an initially stressed conducting medium s discussed by Datta (1986).

The effect of gravity on Rayleigh wave propagation in an elastic solid medium was firstly sidered by Bromwich (1969). Love (1957) studied the influence of gravity on superficial waves l showed that the Rayleigh wave velocity is effected by the gravity field. Biot (1965) showed influence of gravity on Rayleigh waves, assuming that the force of gravity which creates a e of initial stress of hydrostatic nature and the medium to be incompressible. Datta (1986) estigated the effect of gravity on Rayleigh wave propagation in a homogeneous isotropic stic solid medium.

In the present paper, the effect of initial stress, magnetic field and gravity on Rayleigh ve propagation in an elastic solid medium has been discussed using the wave equations isfied by ϕ and ψ. The result obtained is in agreement with the corresponding result when gravity field, the initial stress and the magnetic field are neglected.

2. Formulation of the Problem. Let us consider an electrically conducting charge plied to free homogeneous elastic solid medium under an initial stress, permeated by a uniform gnetic field and the influence of gravity. We take a set of orthogonal cartesian axes $ox_1\ x_2\ x_3$, origin o being any point of the boundary and ox_3 pointing normally to the medium and the axes taken positive along the direction of Rayleigh wave propagation. At a great distance m the centre of the disturbance, the wave is then two dimensional and is polarised in the x_3) plane, by which we mean that the components of displacements in the x_1 and x_3 ection : viz, U_1 and U_3 are not zero.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

If g is the acceleration due to gravity, then the components of body forces are $0, z = -g$. Assuming that, the initial stress field due to gravity is hydrostatic, the sta initial stress τ_{ij} are given by (Datta 1986)

$$\tau_{11} = \tau_{33} = \tau; \quad \tau_{13} = 0,$$

where τ is a function of depth.

If ρ is the density of the medium, the equilibrium conditions of the initial stress are as follows (Dutta 1986)

$$\frac{\partial \tau}{\partial x_1} = 0, \quad \frac{\partial \tau}{\partial x_3} - \rho g = 0.$$

Using the Lame potential's ϕ and ψ, then the displacement vector can be writt the form:

$$U = \Delta\phi + \text{curl}\psi,$$

where $\psi = (0, -\psi, 0)$.

The components of the displacements U_1 and U_3 are given by

$$U_1 = \frac{\partial \phi}{\partial x_1} + \frac{\partial \psi}{\partial x_3}; \quad U_3 = \frac{\partial \phi}{\partial x_3} - \frac{\partial \psi}{\partial x_1},$$

where ϕ and ψ are functions of the coordinates x_i and time t.

From (3), we have

$$\nabla^2\phi = \Delta, \quad \nabla^2\psi = \frac{\partial U_1}{\partial x_3} - \frac{\partial U_3}{\partial x_1},$$

where $\Delta = \dfrac{\partial U_1}{\partial x_1} + \dfrac{\partial U_3}{\partial x_3}$.

The equations of motion for a perfectly conducting elastic solid under hydros tension or compression and the influence of gravity in a uniform field are given by (Dutta

$$(\mu - p_o)\nabla^2 U_1 + (\lambda + \mu)\frac{\partial \Delta}{\partial x_1} - \rho g \frac{\partial U_3}{\partial x_1} = \rho \frac{\partial^2 U_1}{\partial t^2},$$

$$(\mu - p_o)\nabla^2 U_3 + (\lambda + \mu)\frac{\partial \Delta}{\partial x_3} + k\left(\frac{\partial^2 U_3}{\partial x_1^2} - \frac{\partial^2 U_1}{\partial x_1 \partial x_3}\right) + \rho g \frac{\partial U_1}{\partial x_1} = \rho \frac{\partial^2 U_3}{\partial t^2}$$

where ρ is the density of the elastic solid, $-p_o$ is the hydrostatic tension or compression (ten when $p_o < 0$ and compression when $p_o > 0$), λ, μ are the Lame constants and

$$k = \mu_c H_o^2,$$

where μ_c is the magnetic permeability and H_o, the intensity of the uniform magnetic f parallel to x_1 axes.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Substituting from (3) into (4) and (5), one can obtain

$$(\lambda + 2\mu - p_o)\nabla^2\phi - k\frac{\partial^2\psi}{\partial x_1 \partial x_3} + \rho g\frac{\partial\psi}{\partial x_1} = \rho\frac{\partial^2\phi}{\partial t^2}, \tag{7}$$

$$(\mu - p_o)\nabla^2\psi + k\frac{\partial\psi}{\partial x_1^2} - \rho g\frac{\partial\phi}{\partial x_1} = \rho\frac{\partial^2\psi}{\partial t^2}. \tag{8}$$

2. Solution of the Problem. To solve the equations (7) and (8), we put

$$\phi = f(x_3)e^{i\alpha(x_1 - Ct)}, \tag{9}$$

$$\psi = h(x_3)e^{i\alpha(x_1 - Ct)}. \tag{10}$$

The above system will lead us to a particular solution corresponding to Rayleigh waves, wave length $\frac{2\Pi}{\alpha}$ travelling with the speed C.

Substituting (9) and (10) in (7) and (8), we get

$$\left[(\lambda + 2\mu - p_o)(D^2 - \alpha^2) + \rho\alpha^2 C^2\right]f + i\alpha(\rho g - kD)h = 0, \tag{11}$$

$$\left[(\mu - p_o)(D^2 - \alpha^2) + \alpha^2(\rho C^2 - k)\right]h - i\alpha\rho g\, f = 0, \tag{12}$$

ere the D^n operator is defined by $D^n = \frac{d^n}{dx_3^n}$, $(n = 1, 2, \ldots)$

From (11) and (12), one can obtain

$$(D^4 + a_1 D^2 - a_2 D + a_3)h = 0, \tag{13}$$

ere

$$a_1 = \alpha^2\left[\frac{\rho C^2 - k}{\mu - p_o} - 1\right] + \alpha^2\left[\frac{\rho C^2}{\lambda + 2\mu - p_o} - 1\right], \quad a_2 = \frac{\alpha^2\rho g k}{(\mu - p_o)(\lambda + 2\mu - p_o)},$$

$$a_3 = \alpha^4\left[\frac{\rho C^2 - k}{\mu - p_o} - 1\right]\left[\frac{\rho C^2}{\lambda + 2\mu - p_o} - 1\right] - \frac{\alpha^2\rho^2 g^2}{(\mu - p_o)(\lambda + 2\mu - p_o)}. \tag{14}$$

The solution of the equation (13) is

$$\begin{aligned} h = {} & B_1 e^{-\alpha(m+is)x_3} + B_2 e^{-\alpha(m+ir)x_3} + B_3 e^{-\alpha(m-is)x_3} \\ & + B_4 e^{-\alpha(m-ir)x_3} + B_5 e^{\alpha(m+is)x_3} + B_6 e^{\alpha(m+ir)x_3} \\ & + B_7 e^{\alpha(m-is)x_3} + B_8 e^{\alpha(m-ir)x_3}, \end{aligned} \tag{15}$$

ere

$$m = \frac{1}{2\alpha}(2\beta - a_1)^{1/2}, s = \frac{1}{2\alpha}\left[(2\beta + a_1) + 4(\beta^2 - a_3)^{1/2}\right]$$

$$r = \frac{1}{2\alpha}\left[(2\beta + a_1) - 4(\beta^2 - a_3)^{1/2}\right]^{1/2},$$

$$\begin{aligned} \beta = {} & \frac{a_1}{6} + \left(\frac{1}{432}\right)^{1/3}\left[(2a_1^3 - 72a_1a_3 + 27a_2^2) + \left[(2a_1^3 - 72a_1a_3 + 27a_2^2)^2\right.\right. \\ & \left.\left. - 4(a_1^2 + 12a_3)^3\right]^{1/2}\right]^{1/3} + \left(\frac{1}{432}\right)^{1/3}\left[(2a_1^3 - 72a_1a_3 + 27a_2^2)\right. \\ & \left. - \left[(2a_1^3 - 72a_1a_3 + 27a_2^2)^2 - 4(a_1^2 + 12a_3)^3\right]^{1/2}\right], \end{aligned} \tag{16}$$

ere $B_1, B_2, \ldots, B_8$ are constants.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In order that the effect is essentially a surface one, $h(x_3)$ must diminish indefi with the increasing distance from the boundary. This requirement is fulfilled, if $h(x_3)$ co an exponential factor, in which the exponent is real and positive. Hence r, s are taken imaginary, and the constants corresponding to $B_3, B_4, \ldots, B_8$ must vanish.

Therefore the solutions of (7) and (8) are

$$\phi = \left[(A_1 e^{-\alpha(m+is)X_3} + A_2 e^{-\alpha(m+ir)x_3}\right] e^{i\alpha(x1-Ct)},$$
$$\psi = \left[(B_1 e^{-\alpha(m+is)x_3} + B_2 e^{-\alpha(m+ir)x_3}\right] e^{i\alpha(x1-Ct)},$$

where the constants A_1, A_2 are related, respectively, with B_1, B_2 by means of (11) and Equating the coefficients of $e^{-\alpha(m+is)x_3}$ and $e^{-\alpha(m+ir)x_3}$ to zero after substituting from and (18) in (7), we obtain

$$B_1 = (L_1 + in_1)A_1, \quad B_2 = (L_2 + in_2)A_2,$$

where

$$L_1 = \frac{\alpha^2 ks\left[\rho C^2 + (\lambda + 2\mu - p_o)(m^2 - s^2 - 1)\right] - 2\alpha ms(\lambda + 2\mu - p_o)(\rho g + \alpha km)}{(\alpha ks)^2 + (\rho g + \alpha km)^2}$$
$$n_1 = \frac{\alpha(\rho g + \alpha km)\left[\rho C^2 + (\lambda + 2\mu - p_o)(m^2 - s^2 - 1)\right] + 2\alpha^2 kms^2(\lambda + 2\mu - p_o)}{(\alpha ks)^2 + (\rho g + \alpha km)^2},$$
$$L_2 = \frac{\alpha^2 kr\left[\rho C^2 + (\lambda + 2\mu - p_o)(m^2 - r^2 - 1)\right] - 2\alpha mr(\lambda + 2\mu - p_o)(\rho g + \alpha km)}{(\alpha kr)^2 + (\rho g + \alpha km)^2},$$
$$n_2 = \frac{\alpha(\rho g + \alpha km)\left[\rho C^2 + (\lambda + 2\mu - p_o)(m^2 - r^2 - 1)\right] + 2\alpha^2 kmr^2(\lambda + 2\mu - p_o)}{(\alpha kr)^2 + (\rho g + \alpha km)^2}.$$

Hence, the final form of the solution is

$$\phi = \left[(A_1 e^{-\alpha(m+is)x_3} + A_2 e^{-\alpha(m+ir)x_3}\right] e^{i\alpha(x_1-Ct)},$$
$$\psi = \left[(L_1 + in_1)(A_1 e^{-\alpha(m+is)x_3} + (L_2 + in_2)A_2 e^{-\alpha(m+ir)x_3}\right] e^{i\alpha(x_1-Ct)},$$

we shall now apply the boundary conditions, that the plane $x_3 = 0$ is free from stresses,

$$\tau_{13} = \tau_{33} = 0 \quad \text{on } x_3 = 0,$$

where

$$\tau_{31} = \mu\left[2 - \frac{\partial^2\phi}{\partial x_3 \partial x_1} - \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_3^2}\right],$$
$$\tau_{33} = \lambda\nabla^2\phi + 2\mu\left(\frac{\partial^2\phi}{\partial x_3^2} \cdot \frac{\partial^2\psi}{\partial x_3\, \partial x_1}\right).$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In view of (23), (24) and (25), we get

$$\{[(1+m^2-s^2)L_1+2s(1-mn_1)]+i[(1+m^2-s^2)n_1-2m(1-sL_1)]\}$$
$$A_1+\{[(1+m^2-r^2)L_2+2r(1-mn_2)]+i[(1+m^2-r^2)n_2-2m(1-rL_2)]\}$$
$$A_2=0, \tag{26}$$
$$\{[(\lambda+2\mu)(m^2-s^2)-\lambda-2\mu(mn_1+sL_1)]+i[2ms(\lambda+2\mu)+$$
$$2\mu(L_1+sn_1)]\}A_1+[(\lambda+2\mu)(m^2-r^2)-\lambda-2\mu(mn_2+rL_2)]+$$
$$i[2mr(\lambda+2\mu)+2\mu(mL_2-rn_2)]\}A_2=0. \tag{27}$$

Eliminating A_1 andA_2 from (26) and (27), we obtain

$$\{[(1+m^2-s^2)L_1+2s(1-mn_1)]\ [(\lambda+2\mu)(m^2-r^2)-\lambda-2\mu(,mn_2+rL_2)]$$
$$-[(1+m^2-s^2)n_1-2m(1-sL_1)][2mr(\lambda+2\mu)+2\mu(mL_2-rn_2)]\}+i$$
$$\{[(1+m^2-s^2)n_1-2m(1-sL_1)]\ [(\lambda+2\mu)(m^2-r^2)-\lambda-2\mu(mn_2+rL_2)]$$
$$+[(1+m^2-s^2)L_1+2s(1-mn_1)]\ [2mr(\lambda+2\mu)+2\mu(mL_2-rn_2)]\}$$
$$=\{[(1+m^2-r^2)L_2+2r(1-mn_2)]\ [(\lambda+2\mu)(m^2-s^2)-\lambda-2\mu(mn_1+$$
$$sL_1)]-[(1+m^2-r^2)n_2-2m(1-rL_2)]\ [2ms(\lambda+2\mu)+2\mu(mL_1-sn_1)]\}$$
$$+i\{[(1+m^2-r^2)L_2+2r(1-mn_2)]\ [2ms(\lambda+2\mu)+2\mu(mL_1-sn_1)]+$$
$$[(1+m^2-r^2)n_2-2m(1-rL_2)]\ [(\lambda+2\mu)(m^2-s^2)-\lambda-2\mu(mn_1+sL_1)]\},r. \tag{28}$$

Equation (28) determines the velocity C of Rayleigh waves for elastic solid medium under the effect of initial stress, magnetic field and the gravity field.

3. Discussion. The above transcendental equation (28) represents the magneto-elastic Rayleigh surface wave in an initially stressed conducting medium under the gravity field.

In the absence of the gravity field, equation (28) becomes

$$2\delta_o^2\gamma_o^2\left[\left(1-\frac{C^2}{\gamma_o^2}\right)\left(1-\frac{C^2}{\delta_o^2}-\frac{k}{\rho\delta_o^2}\right)\right]^{1/2}\left[\gamma^2k\left(C^2-\frac{k}{\rho}\right)-2\delta^2[k(\delta_o^2+\gamma_o^2)\right.$$
$$\left.-\rho C^2(\delta_o^2-\gamma_o^2)]\right\}=(2\delta^2\gamma_o^2-\gamma^2C^2)\left\{\left(2\delta_o^2-C^2-\frac{k}{\rho}\right)[\rho C^2(\partial_o^2-\gamma_o^2)\right.$$
$$\left.-k\gamma_o^2]+2k\delta_o^2\left(C^2-\delta_o^2+\frac{k}{\rho}\right)\right\}, \tag{29}$$

where

$$\gamma_o^2=\frac{1}{\rho}(\gamma+2\mu-p_o),\quad \delta_o^2=\frac{1}{\rho}(\mu-p_o),\quad \gamma^2=\frac{1}{\rho}(\lambda+2\mu),\quad \delta^2=\frac{\mu}{\rho}.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

When the medium is initially unstressed and non-magnetic, equation (28) takes form

$$\left(2-\frac{C^2}{\delta^2}\right)^2-4\left(1-\frac{C^2}{\gamma^2}\right)^{1/2}+\frac{4g}{\alpha C^2(\gamma^2-\delta^2)}\left\{\left(1-\frac{C^2}{\delta^2}\right)^{1/2}[(\gamma^2+\delta^2)-C^2]\right.$$
$$\left.-\left(1-\frac{C^2}{\gamma^2}\right)^{1/2}\left[(\gamma^2+\delta^2)-\frac{C^2\gamma^2}{\delta^2}\right]\right\}=0. \qquad (3$$

Hence it follows that, when the gravity field is neglected, the Rayleigh surface wa equation given by (29) is in agreement with the corresponding result obtained by Dutta (198

Moreover, equation (30) is in agreement with the corresponding result obtained Datta (1986), where the electromagnetic field and the initial stress are neglected.

References

Biot, M. A. (1940) : *J. Appl. Phys. 11*, 522.

Biot. M. A. (1965) : *Mechanics of Incremental Deformations*, New York.

Bromwich, T. J. I. A. (1969) : *Proc. London Math. Soc., 30*, 98.

Chadwick, P. (1957) : ixth congr. inter. de Mecan. Appl., *7*, 143.

Datta, B. K. (1986) : *Rev. Roum. Sci. Techn. Mec. Appl. 31*, 277.

Datta, B. K. (1986) : *Rev. Roum. Sci. Techn. Mec. Appl. 31*, 369.

Knopoff. L. (1955) : *J. Geophysics, Res. 60*, 441.

Love, A. E. H. (1957) : *Some Problems of Geodynamics*, New York.

Tang, C. P. Yu. S. (1966) : *Zamp, 17*, 766.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
BENHA
EGYPT

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
SOHAG
EGYPT

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
ASWAN
EGYPT

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc., 86, 249–252 (1994)

FURTHER RESULTS ON NON-ARCHIMEDEAN σ BARRELLED IN LOCALLY CONVEX SPACES

MIGUEL CALDAS CUEVA

(Received 22 January 1993; revision received 4 November 1993)

1. Abstract. The purpose of this article is to develop a non-archimedean version of σ-barrelled locally convex spaces, analogous to the classical linear theory of locally convex spaces over C or R. Their relation with other non-archimedean significant classes of locally convex spaces are established and separating examples are given.

Introduction. The non-archimedean analogues of barrelled and bornological locally convex spaces over R or C were introduced by A. F. Monna (1962) and J. Van Tiel (1965). In the present article we define the larger class of non-archimedean σ-barrelled locally convex spaces and, as main result, is to prove a characterization of this class of spaces over a spherically complete field, through a non-archimedean version of the closed graph theorem (Theorem 2).

We shall adopt the notation and terminology of ((Caldas 1992), (Caldas 1992), (Tiel 1965) and (Tiel 1966)). Let K be a field with a non-trivial non-archimedean (n.a.) valuation and (E, τ_E) a locally K-convex space (i.e., n.a.-locally convex spaces over K) endowed with the locally K-convex Hausdorff topology τ_E. As in (Tiel 1966) if A is a subset of E, the pseudo-polar A^p (resp. pseudobipolar A^{pp}) of A is defined as $A^p = \{g \in E' : |g(A)| < 1\}$ (resp. $A^{pp} = \{x \in E : |A^p(x)| < 1\}$. We have $A = A^{pp}$ if and only if A is K-convex and closed ((Tiel 1966) Proposition 2)). For basic facts on locally K-convex spaces we refer to (Tiel 1965) and (Tiel 1966).

DEFINITION 1. (Caldas 1992) A locally K-convex space (E, τ_E) is said to be σ-K-barrelled if each $\sigma(E', E)$-bounded sequence in E' is equicontinuous.

It is known (n.a. Banach-Mackey Theorem) that, if E is a quasicomplete Hausdorff locally K-convex space, then every subject of its duel E' which is bounded for $\sigma(E', E)$ is also bounded for $b(E', E)$. The following proposition shows that the same property holds if E is σ-K-barrelled.

PROPOSITION 1. Let E be a σ-K-barrelled space. Then every $\sigma(E', E)$-bounded subset of E' is $b(E', E)$-bounded.

Proof. Let B be weakly bounded subset of E', and suppose that B is not strongly bounded. Fix $\mu \in K$ with $|\mu| > 1$. Then the scalar sequence $\lambda_n = \mu^n (n \in N)$ is such that $|\lambda_0| < |\lambda_1| < \ldots < |\lambda_n| < \ldots$ and $\lim_n |\lambda_n| = +\infty$. Since B is not strongly bounded in E', there exists a weakly bounded set $A \subset E$ such that for each $n \in N$, we can find $x_n \in A, y_n \in B$ with $| < x_n, y_n > | > |\lambda_n|^2$. Now consider the sequence $z_n = y_n/\lambda_n (n \in N)$ $E', (z_n : n \in N)$ converges to zero in $(E', \sigma(E', E))$ and hence equicontinuous since E is σ-K-barrelled. This

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

implies that $(z_n : n \in N)$ is strongly bounded. But $| < x_n, y_n > | > |\lambda_n|, n \in N$ which is a contradiction. This completes the proof. Q.E.D.

As consequence of the Proposition 1, we have the following result:

PROPOSITION 2. Let E be a σ-K-barrelled space. Then the following assertions hold

1. E is K-infrabarrelled (Pombo. Jr. 1991) if, and only if, E is K-barrelled (Monna 1962).
2. E is Countably K-infrabarrelled (Caldas 1992) if, and only if, E is countably K-barrelled (Caldas 1992).

Since a metrizable locally K-convex space is K-bornological (Tiel 1965) which, in turn is K-infrabarrelled, we have the following:

PROPOSITION 3. Every metrizable (or even K-bornological) σ-K-barrelled space is k-barrelled.

Now reasoning as in (Caldas 1992) or (Caldas 1992) one can prove the following Proposition 4. Let K be a spherically complete field, and let ψ the family of all $\sigma - k$-barrelled σ-K-infrabarrelled spaces (Caldas 1962), respectively. Then ψ is stable under the formation of

(i) Separated quotients, (ii) arbitrary direct sums, (iii) inductive limits, (iv) completions and (v) arbitrary products.

DEFINITION 2. Let (E, τ_E) be a locally K-convex space.

(a) A closed K-convex and absorbing subset of E is said to be a K-barrel.

(b) A K-barrel T in E is said to be a σ-K-barrel if there is a sequence $(y_n : n \geq 1)\sigma(E', E)$-bounded in E' with $T = \bigcap_{n\geq 1} (y_n)^p$.

THEOREM 1. *Let K be a spherically complete field. For a locally K-convex space (E, τ_E) the following are equivalent:*

(i) Each σ-K-barrel in E is a neighbourhood of zero.

(ii) E is σ-K-barrelled.

Proof. (i) $\rightarrow$ (ii) : If $(y_n : n \geq 1)$ is $\sigma(E', E)$-bounded sequence in E', we can write $(y_n : n \geq 1)^p = [\bigcup_{n\geq 1}(y_n)]^p = \bigcap_{n\geq 1}(y_n)^p$. Thus, if we put $T = \bigcap_{n\geq 1}(y_n)^p$ which is certainly σ-K-barrel in E, then $T^p \supset \bigcup_{n\geq 1}(y_n)^{pp}$ and hence $\bigcup_{n\geq 1}(y_n)^{pp}$ is equicontinuous, because by hypothesis T is a neighbourhood of zero. But then $(y_n : n \geq 1)$ is equicontinuous which proves that E is a σ-K-barrelled space.

(ii) $\rightarrow$ (i) : Let $T = \bigcap_{n\geq 1}(y_n)^p$ be a σ-K-barrel in E where $(y_n : n \geq 1)$ in E' is a $\sigma(E', E)$-bounded sequence. Since E' is σ-K-barrelled, then $(y_n : n \geq 1)$ is equicontinuous. Therefore,

$$T = \bigcap_{n\geq 1}(y_n)^p = \left[\bigcup_{n\geq 1}(y_n)\right]^p = (y_n : n \geq 1)^p$$

is a neighbourhood of zero in E, which proves the theorem. Q.E.D.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In the linear theory of locally convex spaces over R or C it is well known that a locally convex space E is σ-barrelled if, for every separable Banach space H and for every linear mapping $f : E \to F$ (F being the strong dual of H) whose graph is closed in Ex $(F, \sigma(F, H))$, we have that f is continuous. We will prove the corresponding result in the non-archimedean context.

Before we prove our result, let us recall that

$$1^\infty = \{(\lambda_n : n \geq 1) : \lambda_n \in K \exists M > 0 |\lambda_n| \leq M \forall n\}$$
$$\text{and} \quad C_\infty = \{(\lambda_n : n \geq 1) : \lambda_n \in K \exists n_o \geq 1 (\forall n \geq n_o) \lambda_n = 0\}$$

forms a dual system with respect to the bilinear form $B : 1^\infty \times C_\infty \to K$, defined by $B(\lambda, \gamma) = \Sigma \lambda_n \gamma_n$ (finite sum) for $\lambda = (\lambda_n) \in 1^\infty, \gamma = (\gamma_n) \in C_\infty$.

THEOREM 2. *Let (E, τ_E) be a Housdorff locally K-convex space over K spherically complete, such that every linear mapping from (E, τ_E) to $(1^\infty, \|.\|)$ with $\tau \times \tau_{\|\ \|}$-closed graph is continuous. Then (E, τ_E) is σ-K-barrelled.*

Proof. Let $T = \bigcap_{n \geq 1} (y_n)^p$ be a σ-K-barrel in E, where $(y_n : n \geq 1)$ is a $\sigma(E', E)$-bounded sequence. Then for each $x \in E$, there exists $\lambda \in K - (0)$ such that $x/\lambda \in T$, because T is absorbing. Hence $|y_n(x/\lambda)| < 1$ for each n. Then $\sup_n |y_n(x)| < |\lambda|$. This shows that $(y_n : n \geq 1) \in 1^\infty$. Then we may define a map $\phi_T : E \to 1^\infty$ by taking $\phi_T(x) = (y_n(x) : n \geq 1)$ (if $x \in E$).

Clearly, ϕ_T is linear. Now, we shall prove that ϕ_T is $\tau_E - \sigma(1^\infty, C_\infty)$ continuous. For this purpose, it suffices to prove that for every net $(x_i : i \in I)$ converging to zero in (E, τ_E), the net $(\phi_T(x_i) : i \in I)$ converges to zero in $(1^\infty, \sigma(1^\infty, C_\infty)$. By the definition of $\sigma(1^\infty, C_\infty)$, it is enough to show that, for every $\gamma = (\gamma_n : n \geq 1) \in C_\infty$, the net

$$\left(\sum_{1 \leq k \leq n_o} y_k(x_i) \gamma_k : i \in I \right)$$

converges to zero, when n_o is such that $y_n = 0$ for all $n > n_o$. But this follows from the continuity of $y_k (k = 1, \ldots, n_o)$. Consequently, the graph of ϕ_T is $\tau_E x \sigma(1^\infty, C_\infty)$-closed, hence $\tau_E \tau_{\|\ \|}$-closed, because $\sigma(1^\infty, C_\infty) \leq \tau_{\|\ \|}$. Then by assumption ϕ_T is ${}^\tau E_{\|\ \|}^{-\tau}$ continuous.

Finally, we will prove that $T \supset \phi^{-1}(B)$, where B is the unit ball in $(1^\infty, \|\ .\ \|)$, defined by $B = \{\lambda \in 1^\infty : \|\lambda\| < 1\}$. Really, let $x \in E$ with $x \in \phi^{-1}(B)$: then $\phi_T(x) \in B$. Hence $\|\phi_T(x)\| = \|(y_n(x) : n \geq 1)\| = \sup_n \|y_n(x)\| < 1$ and by definition $x \in (y_n)^p$ for all $n \geq 1$, or equivalently $x \in \bigcap_{n \geq 1} (y_n)^p = T$. This proves that T is a neighbourhood of zero in E. Thus, (E, τ_E) is a σ-K-barrelled space. Q.E.D.

We finish our article with the following example of a σ-K-infrabarrelled space (Caldas 1992) or which is not k-barrelled.

EXAMPLE 1. Let $F = K^N$. For each $n = 1, 2, \ldots$ define the semi-norm P_n on F by $P_n(x) = |\lambda_n| (\forall x = (\lambda_n : n \geq 1) \in F)$. Obviously y_n, P is a n. a.-seminorm on F. Then the

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

metrizable locally K-convex topology generated by the family$(P_n : n \geq 1)$ of n.a.-seminorms is just the product topology.

Consider $E = K^{(N)}$ the K-vector subspace of F (of the sequences $(\lambda_j : j \geq 1)$ such that $\lambda_j = 0$ except for finitely many indices j); endowed with the induced topology. Now, let us prove that the metrizable space E which, in turn is countably-k-infrabarrelled is not K-barrelled. In fact, let $T = \{(\lambda_j : j \geq 1) \in E; \max_{j\geq 1} |\lambda_j| \leq 1\}$. T is K-convex and absorbent, since, if $x = (\lambda_j : j \geq 1) \in E$ and $\mu \in K$ is such that $|\mu| \max_{j\geq 1} |\lambda_j|$, then $\mu^{-1}x \in T$. T is closed since $T = \bigcap_{n\geq 1} \{x \in E; P_n(x) \leq 1\}$. Therefore, T is a K-barrel in E (Definition 2 (a)). Suppose that T is a neighbourhood of zero in E. Then there exists $\epsilon > 0, r \geq 1, n_1 < \ldots < n_r \in N - \{0\}$ such that

$$W = \{x \in E; P_{n_1}(x) \leq \epsilon, \ldots, P_{n_r}(x) \leq \epsilon\} \subset T. \qquad (*)$$

Define $x = (\lambda_j : j \geq 1) \in E$ by $\lambda_j = 0$ if $j \neq n_r + 1; \lambda_{n_r+1} \in K$ with $|\lambda_{n_r+1}| > 1$. Obviously $x \in W$ and $x \notin T$; which is a contradiction with $(*)$. Then T is not a neighbourhood of zero, and this implies (by Theorem 1) that E is not K-barrelled.

On the other hand, E is not σ-K-barrelled, for otherwise E would be K-barrelled, by Proposition 2. Therefore E is also an example of a σ-K-infrabarrelled space which is not σ-K-barrelled.

References

Caldas, M. (1992) : Stability of Barrelledness and Infrabarrelledness in Locally K-convex Spaces. *Bull. Cal. Math. Soc. 84,* 97.

Caldas, M. (1992) : Uma Generalizacao Nao-Arquimediana do Teorema de Mahowald para Espacos d-Tonelados. *Portug. Math. 49,* 241.

Monna, A. F. (1963) : Sur le Theoreme de Banach-Steinhaus. *Proc. Kon. Ned. Akad. Van Wetensch A 66,* 121.

Monna, A. F. (1962) : Espaces Vectoriels Topologiques Sur Un Corps Value. *Proc. Kon. Ned. Akad. Van Wetensch A 65,* 351.

Pombo Jr., D. (1991) : On Infrabarrelled Locally K-convex Spaces. *Indian Nat. Sc. Ac. 22,* 575.

Van Tiel, J. (1965) : Espaces Localement K-Convexes. *Proc. Kon. Ned. Akad. Van Wetensch A 68,* 249.

Van Tiel, J. (1966) : Ensembles Pseudo-polaires Dans les Espaces Localement K-convexes. *Proc. Kon. Ned. Akad. Van. Wetensch A 69,* 369.

DEPARTMENTO DE MATEMÁTICA APLICADA-IMUFF
UNIVERSIDADE FEDERAL FLUMINENSE
RUA SAO PAULO S/N° CEP: 24210
NITEROI-RJ
BRASIL

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc., 86, 253–266 (1994)

OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF A CLASS OF NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS OF THIRD ORDER

N. PARHI AND P. DAS

(*Received 11 August 1992; revision received 24 August 1993*)

Abstract. Oscillatory and asymptotic behaviour of solutions of a class of nonlinear and nonhomogeneous functional-differential equations of third order have been studied in this paper. It has been shown that the behaviour of solutions of such equations differ from that of corresponding ordinary differential equations.*

1. In this paper we consider nonlinear third order functional differential equations of the form

$$y'''(t) + a(t)y''(t) + b(t)y'(t) + c(t)F(y(g(t))) = 0 \tag{1.1}$$

$$y'''(t) + a(t)y''(t) + b(t)y'(t) + c(t)y^{\gamma}(g(t)) = 0 \tag{1.2}$$

$$(r(t)y''(t))' + q(t)y'(t) + p(t)y^{\gamma}(g(t)) = 0 \tag{1.3}$$

and $$(r(t)y''(t))' + p(t)y^{\gamma}(g(t)) = f(t), \tag{1.4}$$

where a, b, c, p, q, r and $f \in C([\sigma, \infty), R), \sigma \in R$, such that $r(t) > 0, g \in C^1([\sigma, \infty), R)$ such that $g(t) \leq t, g(t) > 0, g'(t) \geq 0$ and $g(t) \to \infty$ as $t \to \infty, F \in C(R, R)$ such that $F(u)/u > \beta > 0$ for $u \neq 0$ and $\gamma > 0$ is a quotient of odd integers. Section 2 deals with the asymptotic behaviour of nonoscillatory solutions of (1.1) and (1.2). Sufficient conditions have been obtained in Section 3 so that bounded solutions of (1.3) are oscillatory or tend to zero as $t \to \infty$. In Section 4, sufficient conditions have been obtained for oscillation of all solutions/all bounded solutions of (1.4).

We assume that under the initial conditions $y(t) = \psi(t), t \leq t_0$ and $y^{(j)}(t_0) = y_j^0$, $j = 1, 2$, where $t_0 \in [\sigma, \infty)$, the equations (1.1)-(1.4) admit solutions which can be continued to $[t_0, \infty)$. A function $y \in C([\sigma, \infty)R)$ is said to be oscillatory if it has arbitrarily large zeros. Otherwise, it is said to be nonoscillatory. It is interesting to note that there are delay-differential equations which admit both oscillatory and non-oscillatory solutions. For example,

$$y'''(t) + y''(t) + y'(t) - y(t - \pi) = 0 \tag{1.5}$$

* This work was supported by the Council of Scientific and Industrial Research, India, under Grant No. 9/297 (35/90/EMR-1).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

admits an oscillatory solution $y_1(t) = \sin t$. The characteristic equation $\phi(\lambda) \equiv \lambda^3 + \lambda^2 + \lambda - e^{-\pi\lambda} = 0$ of (1.5) has the properties $\phi(0) < 0$ and $\phi(\lambda) \to \infty$ as $\lambda \to \infty$. So there exists a $\lambda_1 > 0$ such that $\phi(\lambda_1) = 0$ and $y_2(t) = e^{\lambda_1 t}$ is a nonoscillatory solution of (1.5). The nonhomogeneous delay-differential equation

$$y'''(t) - e^{\pi/2} y\left(t - \frac{\pi}{2}\right) = \left(e^{\pi/2} - 1\right) \cos t, t \geq 0$$

admits an oscillatory solution $y_1(t) = \sin t$ and a nonoscillatory solution $y_2(t) = \sin t + e^t$.

In the literature there are many papers, for example see (Grace and Lalli 1985, Kartsatos and Manougian 1974, Kusano and Onose 1973, 1974), which deal with oscillatory and asymptotic behaviour of solutions of functional differential equations of arbitrary order. Our work is different from theirs in the sense that either the conditions assumed by them are not satisfied by the equations we consider or the type of equations they consider are different from ours. The results we obtain here are different from those concerning ordinary differential equations of third order due to the presence of the delay. The following example shows how the presence of delay in the differential equation changes the nature of solutions. From a result in (Ladas, Sficas and Stavroulakis 1984, Theorem 3) it follows that all solutions of $y'''(t) + y(t - \tau) = 0$, $\tau > 0$ are oscillatory if and only if $\tau e > 3$. But the corresponding ordinary differential equation $y''' + y = 0$ admits nonoscillatory solution $y_1(t) = e^{-t}$ and oscillatory solutions $y_2(t) = e^{t/2} \cos \frac{\sqrt{3}}{2} t$ and $y_3(t) = e^{t/2} \sin \frac{\sqrt{3}}{2} t$. Some of our results are also valid for advanced differential equations.

2. This section deals with the asymptotic behaviour of non-oscillatory solutions of (1.1) and (1.2).

THEOREM 2.1. *Suppose that $a(t) \geq 0$ is continuously differential such that $a'(t) \leq 0$, $b(t) \leq 0, c(t) > 0$ and*

$$\int_\sigma^\infty \left[\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + \beta c(t) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t)\right)^{3/2}\right] dt = \infty. \tag{2.1}$$

Then every nonoscillatory solution of (1.1) tends either to $\pm\infty$ or to zero as $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (1.1) on $[t_0, \infty), t_0 \geq \sigma$. Suppose that $y(t)$ is ultimately positive. The case when $y(t)$ is ultimately negative may be treated similarly. So there exists a $t_1 \geq t_0$ such that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_1$. Since $y'(t)$ is a solution of $z''(t) + a(t)z'(t) + b(t)z(t) = -c(t)F(y(g(t)))$, that is, of

$$(R(t)z'(t))' + Q(t)z(t) = -P(t)F(y(g(t))), \tag{2.2}$$

where $R(t) = \exp\left(\int_{t_1}^t a(s)\, ds\right), Q(t) = b(t)R(t)$ and $P(t) = c(t)R(t)$, and since all solutions of (2.2) are nonoscillatory (cf. Proposition 2.1, Parhi and Nayak 1984), $y'(t) > 0$ or < 0 for $t \geq t_2 \geq t_1$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Suppose that $y(t) > 0$ for $t \geq t_2$. We claim that $\lim_{t\to\infty} y(t) = \infty$. If not, $\lim_{t\to\infty} y(t) = K$, where $0 < K < \infty$. From the continuity of F we have

$$\lim_{t\to\infty} \frac{F(y(g(t)))}{y(t)} = \frac{F(K)}{K} > \beta.$$

Hence there exists a $T > t_2$ such that, for $t \geq T$, $\dfrac{F(y(g(t)))}{y(t)} > \beta$. Clearly, $u(t) = y'(t)/y(t)$, for $t \geq T$, a solution of $u''(t) + 3u(t)u'(t) + a(t)u'(t)$

$$\begin{aligned}
&= -\left[u^3(t) + a(t)u^2(t) + b(t)u(t) + c(t)\frac{F(y(g(t)))}{y(t)}\right] \\
&\leq -\left[u^3(t) + a(t)u^2(t) + b(t)u(t) + \beta c(t)\right] \\
&\leq -\left[\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + \beta c(t) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t)\right)^{3/2}\right]
\end{aligned}$$

because

$$\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + \beta c(t) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t)\right)^{3/2}$$

is the minimum of $(u^3(t) + a(t)u^2(t) + b(t)u(t) + \beta c(t))$ for $u(t) \geq 0$ at

$$u(t) = \frac{1}{3}\left(-a(t) + (a^2(t) - 3b(t))^{1/2}\right).$$

Integrating the above inequality from T to t and taking the limit as $t \to \infty$, we get $\lim_{t\to\infty} u'(t) = -\infty$. This in turn implies that $u(t) < 0$ for large t, a contradiction. Hence $\lim_{t\to\infty} y(t) = \infty$.

Next suppose that $y'(t) < 0$ for $t \geq t_2$. So $\lim_{t\to\infty} y(t)$ exists. If possible, let $\lim_{t\to\infty} y(t) = \lambda$, where $0 < \lambda < \infty$. Clearly, $A^2(t) - B^2(t) \leq 0$ for $t \geq \sigma$, where

$$A(t) = \frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} \quad \text{and} \quad B(t) = \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t)\right)^{3/2}$$

So $A(t) - B(t) \leq 0$ for $t \geq \sigma$. Consequently, from (2.1) it follows that

$$\int_\sigma^\infty c(t)\,dt = \infty. \tag{2.3}$$

As $y(t)$ satisfies

$$(R(t)y''(t))' + Q(t)y'(t) + P(t)F(y(g(t))) = 0,$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

then integrating (2.3) from t_2 to t, we obtain

$$R(t)y''(t) \leq R(t_2)y''(t_2) - \frac{\lambda\beta}{2}\int_{t_2}^{t} P(s)\,ds \leq R(t_2)y''(t_2) - \frac{\lambda\beta}{2}\int_{t_2}^{t} c(s)\,ds.$$

This in turn implies that $y''(t) < 0$ for large t. Consequently, $y(t) < 0$ for large t, a contradiction. Hence $\lim_{t\to\infty} y(t) = 0$.

This completes the proof of the theorem.

THEOREM 2.2. *If $a(t) \geq 0$ is continuously differentiable such that $a'(t) \leq 0, b(t) \leq 0$, $c(t) > 0$ and*

$$\int_{\sigma}^{\infty}\left[\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + \mu c(t) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t)\right)^{3/2}\right] dt = \infty$$

for all $\mu > 0$, then every nonoscillatory solution of (1.2) tends either to $\pm\infty$ to to zero as $t \to \infty$.

The proof is similar to that of Theorem 2.1 and hence is omitted.

REMARK. (i) If conditions of Theorem 2.1 are satisfied, then every nonoscillatory solution of

$$y''' + a(t)y'' + b(t)y' + c(t)F(y) = 0 \tag{2.4}$$

tends to zero as $t \to \infty$ (Parhi and Das 1993). It is clear that this difference in behaviour between (1.1) and (2.4) is due to the deviation in the argument in (1.1).

(ii) Suppose that the conditions of Theorem 2.2 are satisfied. Then every nonoscillatory solutions of

$$y''' + a(t)y'' + b(t)y' + c(t)y^{\gamma} = 0. \tag{2.5}$$

tends to zero as $t \to \infty$ (Parhi and Das 1993). In this case also the difference in behaviour of (1.2) and (2.5) is due to the deviation in the argument in (1.2).

Following examples illustrate above theorems and strengthen our remakrs.

EXAMPLE 1. Consider

$$y'''(t) + ay''(t) + by'(t) + cy(t-\tau) = 0, \tag{2.6}$$

where a, b, c and τ are reals such that $a > 0, b < -(a+1)$,

$$c > 2\left|\frac{2a^3}{27} - \frac{ab}{3} - \frac{2}{3\sqrt{3}}\left(\frac{a^2}{3} - b\right)^{3/2}\right|$$

and $\tau = -\log\left(\frac{1+a+b}{-c}\right)$. Clearly, all the assumptions of Theorem 2.1 are satisfied and $y(t) = e^{-t}$ is a nonoscillatory solution of (2.6).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

EXAMPLE 2. The differential equation with deviating argument

$$y'''(t) + ay''(t) + by'(t) + cy(t-\tau) = 0,$$

where $a = \frac{1}{4}, b = -\frac{1}{4}, c > 2\left|\frac{2a^3}{27} - \frac{ab}{3} - \frac{2}{3\sqrt{3}}\left(\frac{a^2}{3} - b\right)^{3/2}\right|$ and $\tau = -\log 2c$, admits a nonoscillatory solution $y(t) = e^{-t}$ which tends to zero as $t \to \infty$. It is easy to see that all the conditions of Theorem 2.1 are satisfied.

EXAMPLE 3. The equation $y'''(t) - 2y'(t) + e^{t/2}y(t - \frac{t}{2}) = 0$ illutrates Theorem 2.2. Clearly, $y(t) = e^t$ is a nonoscillatory solution of the equation which tends to infinity as $t \to \infty$.

In the following we state some results from (Parhi and Das 1993) for our use in the sequel.

THEOREM 2.3. *If $a \in C^2([\sigma,\infty), R)$ and $b \in C^1([\sigma,\infty), R)$ such that $a(t) \le 0, b(t) \le 0$, $c(t) > 0, b(t) - a'(t) \le 0$ and*

$$\int_\sigma^\infty \left[\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + c(t) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t) + a'(t)\right)^{3/2}\right] dt = \infty,$$

then

$$y''' + a(t)y'' + b(t)y' + c(t)y = 0 \tag{2.7}$$

is oscillatory.

THEOREM 2.4. *Suppose that $a \in C^2([\sigma,\infty), R)$ and $b \in C^1([\sigma,\infty), R)$ such that $a(t) \le \alpha < 0$ (or $\int_\sigma^\infty a(s)\,ds > -\infty$), $b(t) \le 0$ and $c(t) > 0$. Then (2.7) is oscillatory if and only if its every nonoscillatory solution tends to zero as $t \to \infty$.*

THEOREM 2.5. *Suppose that $a \in C^2([\sigma,\infty), R)$ and $b \in C^1([\sigma,\infty), R)$ such that $a(t) \le \alpha < 0$ (or $\int_\sigma^\infty a(t)\,dt > -\infty$), $b(t) \le 0, c(t) > 0, b(t) - a'(t) \le 0$ and*

$$\int_\sigma^\infty \left[\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + \beta c(t) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t) + a'(t)\right)^{3/2}\right] dt = \infty. \tag{2.8}$$

Then every nonoscillatory solution of (1.1) tends either to $\pm\infty$ or to zero as $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (1.1) on $[t_0,\infty), t_0 \ge \sigma$. We assume that there exists a $t_1 \ge t_0$ such that $y(t) > 0$ and $y(g(t)) > 0$ for $t \ge t_1$. The case when $y(t)$ is ultimately negative may be treated in a similar way. We proceed as in Theorem 2.1 to conclude that $y'(t) > 0$ or < 0 for $t \ge t_2 \ge t_1$.

Let $y'(t) > 0$ for $t \ge t_2$. If possible suppose that $\lim_{t\to\infty} y(t) = K$, where $0 < K < \infty$. Proceeding as in Theorem 2.1, we obtain

$$u''(t) + 3u(t)u'(t) + a(t)u'(t) \le -\left[u^3(t) + a(t)u^2(t) + b(t)u(t) + \beta c(t)\right],$$

where $u(t) = y'(t)/y(t)$. Integrating the above inequality from T to t and using the argument employed in Theorem 2.1 we obtain

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$u'(t) \le u'(T) - \frac{3}{2}u^2(t) + \frac{3}{2}u^2(t) - a(t)u(t) - \int_T^t \left[u^3(s) + a(s)u^2(s) + (b(s) - a'(s))u(s) + \beta c(s)\right]$$

$$\le u'(T) + \frac{3}{2}u^2(T) + \frac{1}{6}a^2(t) - \int_T^t \left[\frac{2a^3(s)}{27} - \frac{a(s)b(s)}{3} + \frac{a(s)a'(s)}{3} + \beta c(s) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(s)}{3} - b(s) + a'(s)\right)^{3/2}\right]$$

$$\le u'(T) + \frac{3}{2}u^2(T) + \frac{1}{6}a^2(T) - \int_T^t \left[\frac{2a^3(s)}{27} - \frac{a(s)b(s)}{3} + \beta c(s) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(s)}{3} - b(s) + a'(s)\right)^{3/2}\right]$$

Hence $u'(t) \to -\infty$ as $t \to \infty$. This in turn implies that $u(t) < 0$ for large contradiction. Thus, $\lim_{t\to\infty} y(t) = \infty$.

Next suppose that $y'(t) < 0$ for $t \ge t_2$. We claim that $\lim_{t\to\infty} y(t) = 0$. If not, $\lim_{t\to\infty} y(t)$ λ, where $0 < \lambda < \infty$. Clearly, $y(t)$ is a nonoscillatory solution of the third order ordin differential equation

$$z''' + a(t)z'' + b(t)z' + c(t)\frac{F(y(g(t)))}{y(t)}z = 0.$$

Since, as in Theorem 2.1, $\dfrac{F(y(g(t)))}{y(t)} \ge \beta$ for large t, from (2.8) it follows that

$$\int_\sigma^\infty \left[\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + c(t)\frac{F(y(g(t)))}{y(t)} - \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t) + a'(t)\right)^{3/2}\right] dt = \infty.$$

Hence by Theorem 2.3, (2.9) admits an oscillatory solution. Consequently, from T orem 2.4, it follows that every nonoscillatory solution of (2.9) tends to zero as $t \to \infty$. particular, $y(t) \to 0$ as $t \to \infty$, a contradiction. Thus, $\lim_{t\to\infty} y(t) = 0$.

Hence the proof of the theorem is completed.

THEOREM 2.6. *Let $a \in C^2([\sigma,\infty)R)$ and $b \in C^1([\sigma,\infty),R)$ such that $a(t) \le \alpha$ $\left(or \int_\sigma^\infty a(t)\,dt > -\infty\right)$, $b(t) \le 0, c(t) > 0$, $b(t) - a'(t) \le 0$ and*

$$\int_\sigma^\infty \left[\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + \mu c(t) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t) + a'(t)\right)^{3/2}\right] dt = \infty \qquad (2$$

for all $\mu > 0$. Then every nonoscillatory solution of (1.2) tends either to $\pm\infty$ or to zer $t \to \infty$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The proof of this theorem proceeds in the lines of that Theorem 2.5 and hence is omitted.

REMARK. If conditions of Theorem 2.6 was satisfied with the assumption $\int_{\sigma}^{\infty} a(t)\,dt > -\infty$, then every non-oscillatory solution of (2.5) with $\gamma \geq 1$ tends to zero as $t \to \infty$ (Parhi and Das 1993). Clearly, this difference is due to the deviation in the argument in (1.2).

EXAMPLE 1. The equation $y'''(t) - y''(t) - y'(t) + e^{t/2} y\left(t - \frac{t}{2}\right) = 0$ admits a nonoscillatory solution $y(t) = e^t$ which tends to ∞ as $t \to \infty$. This example illustrates Theorem 2.6 with the assumption $a(t) \leq \alpha < 0$.

2. The example 3 following Theorem 2.2 illustrates Theorem 2.6 with the other assumption $\int_{\sigma}^{\infty} a(t)\,dt > -\infty$.

REMARK. Theorems 2.1, 2.2, 2.5 and 2.6 remain true if $g(t) \geq t$. However, the following theorem holds only when $g(t) \geq t$.

THEOREM 2.7. *Let $g(t) \geq t$. Suppose that $a \in C^2([\sigma, \infty), R)$ and $b \in C^1([\sigma, \infty), R)$ such that $a(t) \leq 0, b(t) \leq 0, c(t) > 0$, $b(t) - a'(t) \leq 0$ and (2.8) holds. Then every bounded solution of (1.1) which has a zero is oscillatory.*

Proof. Let $y(t)$ be a bounded solution of (1.1) on $[\sigma, \infty)$ such that $y(t_0) = 0, t_0 \geq \sigma$. We claim that $y(t)$ is oscillatory. If not, $y(t)$ is ultimately positive or negative. We assume that $y(t)$ is ultimately positive. The case when $y(t)$ is ultimately negative may be treated in a similar manner. So $y(t) > 0$ for $t > t_0$. We may assume t_0 to be the last zero of $y(t)$ if it has more than one zero. Since $g(t) \geq t$, then $y(g(t)) > 0$ for $t > t_0$. We may proceed as in Theorem 2.1 to conclude that $y'(t) > 0$ or < 0 for $t \geq t_1 \geq t_0$.

If $y'(t) > 0$ for $t \geq t_1$, then $\lim_{t \to \infty} y(t) = K$, where $0 < K < \infty$. Proceeding as in the proof of Theorem 2.5, we arrive at a contradiction.

Thus, $y'(t) < 0$ for $t \geq t_1 \geq t_0$. As $y(t) > 0$ for $t > t_0$, then $t_1 > t_0$ and there exists a $t_2, t_0 < t_2 < t_1$, such that $y'(t_2) = 0$ and $y'(t) < 0$ for $t > t_2$ (We may take t_2 to be the last zero of $y'(t)$ if it has more than one zero to the right of t_0). Now multiplying (2.3) through by $y'(t)$ and integrating the resulting identity from t_2 to $t (t_2 < t)$, we obtain

$$R(t)y'(t)y''(t) = \int_{t_2}^{t} R(s)(y''(s))^2\,ds - \int_{t_2}^{t} Q(s)(y'(s))^2\,ds - \int_{t_2}^{t} P(s)y'(s)F(y(g(s)))\,ds > 0.$$

Hence $y''(t) < 0$ for $t > t_2$. Consequently, $y(t) < 0$ for large t, a contradiction.

Thus, the theorem is proved.

REMARK. Suppose that the conditions of Theorem 2.7 are satisfied with $g(t) = t$. Then every solution of (2.4) which has a zero is oscillatory (Parhi and Das 1993). This difference is due to the change in the argument of y in (1.1).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

THEOREM 2.8. *Suppose that the conditions of Theorem 2.7 are satisfied with* replaced by (2.10). *Then every bounded solution of* (1.2) *which has a zero is oscillatory.*

The proof is similar to that of Theorem 2.7 and hence is omitted.

3. In this section we study oscillatory and asymptotic behaviour of bounded solu of (1.3) and some of its particular cases.

LEMMA 3.1. *Suppose that* $p(t) \geq 0$ *or* ≤ 0 *and*

$$(r(t)z')' + q(t)z = 0$$

is nonoscillatory. If $y(t)$ *is a nonoscillatory solution of* (1.3), *then there exists a* $t_0 \in [$ *such that* $y(t)y'(t) > 0$ *or* $y(t)y'(t) < 0$ *for* $t \geq t_0$).

The proof follows from a result due to Keener (1971) and hence is omitted.

THEOREM 3.2. *Suppose that* $p(t) \geq 0, q(t) \geq 0, q'(t) \leq 0,$

$$\int_\sigma^\infty \frac{dt}{r(t)} = \infty, \quad \int_\sigma^\infty p(t)\, dt = \infty$$

and (3.1) *is nonoscillatory. Then every solution of* (1.3) *is oscillatory or tends to ze* $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (1.3) on $[t_0, \infty), t_0 \geq \sigma$. Wit any loss of generality, we may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_1 \geq t_0$. I Lemma 3.1 it follows that $y'(t) > 0$ or < 0 for $t \geq t_2 \geq t_1$. If $y'(t) > 0$ for $t \geq t_2$, integrating (1.3) from t_2 to t, we obtain $r(t)y''(t) \leq r(t_2)y''(t_2) - y^\gamma(g(t_2)) \int_{t_2}^t p(s)\, ds$. T $r(t)y''(t) \to -\infty$ as $t \to \infty$. Since $\int_\sigma^\infty \frac{dt}{r(t)} = \infty$, then $y'(t) < 0$ for large t, a contradic Hence $y'(t) < 0$ for $t \geq t_2$. If $\lim_{t\to\infty} y(t) = \lambda > 0$, then integrating (1.3) from t_2 to t, we $r(t)y''(t) \leq r(t_2)y''(t_2) + q(t_2)y(t_2) - \lambda^\gamma \int_{t_2}^t p(s)\, ds$. Thus, $y''(t) < 0$ for large t. Conseque $y(t) < 0$ for large t, a contradiction. Hence $\lim_{t\to\infty} y(t) = 0$.

This completes the proof of the theorem.

EXAMPLE. The equation $y'''(t) + \frac{1}{t^3}y'(t)\left(\frac{1}{t} + \frac{1}{t^4}\right)y(t - \log t) = 0,\ t \geq 2$, adm nonoscillatory solution $y(t) = e^{-t}$ which $\to 0$ as $t \to \infty$.

THEOREM 3.3. *Suppose that* $p(t) \geq 0, q(t) \leq 0, q'(t) \leq 0, q(t)$ *is bounded below,*

$$\int_\sigma^\infty \frac{dt}{r(t)} = \infty \quad \text{and} \quad \int_\sigma^\infty p(t)\, dt = \infty.$$

Then every bounded solution of (1.3) is oscillatory or tends to zero as $t \to \infty$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The proof of this theorem proceeds in the lines of that of Theorem 3.2 and hence is omitted.

EXAMPLE. Consider $y'''(t) - 2y'(t) + 4e^{-\pi}y(t-\pi) = 0, \quad t \geq 0.$

This equation admits a bounded oscillatory solution $y(t) = e^{-t}\sin t$ which tends to zero as $t \to \infty$.

THEOREM 3.4. *Suppose that* $p(t) \leq 0$, $\int_\sigma^\infty \frac{dt}{r(t)} = \infty$ *and* $\int_\sigma^\infty p(t)\,dt = -\infty$. *If* $y(t)$ *is a solution of*

$$(r(t)y''(t))' + p(t)y^\gamma(g(t)) = 0, \tag{3.2}$$

then $y(t)$ *is oscillatory or* $|y(t)| \to \infty$ *as* $t \to \infty$.

Proof. If $y(t)$ is oscillatory, then there is nothing to show. Suppose that $y(t)$ is nonoscillatory. Without any loss of generality, we may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_1 \geq \sigma$. Since $(r(t)y''(t))' = -p(t)y^\gamma(g(t)) \geq 0$ for $t \geq t_1$, there exists a $t_2 \geq t_1$ such that $y'(t)$ and $y''(t)$ have no zero for $t \geq t_2$. If $y'(t) > 0$ for $t \geq t_2$, then integrating (3.2) from t_2 to t, we obtain $r(t)y''(t) \geq r(t_2)y''(t_2) - y^\gamma(g(t_2)) \int_{t_2}^t p(s)\,ds$. Thus, $y''(t) > 0$ for large t. Consequently, $y(t) \to \infty$ as $t \to \infty$. Next suppose that $y'(t) < 0$ for $t \geq t_2$. If $y''(t) > 0$ for $t \geq t_2$, then $r(t)y''(t) \geq r(t_2(y''(t_2)$ implies that $y'(t) > 0$ for large t, a contradiction. Hence $y''(t) < 0$ for $t \geq t_2$. This in turn implies that $y(t) < 0$ for large t, a contradiction again.

Hence the theorem is proved.

COROLLARY 3.5. *If the hypotheses of Theorem* 3.4 *are satisfied, then bounded solutions of* (3.2) *are oscillatory.*

EXAMPLE. The equation $y'''(t) - 27e^{2t+3}y^{1/3}(t-1) = 0, \quad t \geq 0$ admits an unbounded nonoscillatory solution $y(t) = e^{3(t+1)}$ and the equation $y'''(t) - y(t - \frac{\pi}{2}) = 0$ admits a bounded oscillatory solution $y(t) = \sin t$.

LEMMA 3.6. *Suppose that* $f \in C^2([\sigma,\infty), R)$ *such that* $f(t) > 0, f'(t) > 0$ *and* $f''(t) < 0$ *for* $t \geq \sigma$. *If* $\tau \in C([\sigma,\infty), R^+)$ *is bounded then* $\lim_{t\to\infty} \frac{f(t-\tau(t))}{f(t)} = 1$.

LEMMA 3.7. *Suppose that* f *and* $g \in C([\sigma,\infty), R)$ *such that* $f(t) > 0, g(t) > 0$ *for* $t \geq \sigma$ *and* $\lim_{t\to\infty} f(t) = 1$. *Then* $\lim_{t\to\infty} \inf f(t)g(t) = \lim_{t\to\infty} \inf g(t)$.

The proofs of above two lemmas are straight-forward and hence are omitted.

LEMMA 3.8. *Suppose that* $Q \in C([\sigma,\infty), R^+)$. *If*

$$z''' - Q(t)z = 0 \tag{3.3}$$

is an oscillatory solution, then nonoscillatory solutions of (3.3) *are unbounded.*

Proof. Let $y(t)$ be an oscillatory solution and $z(t)$ be a nonoscillatory solution of (3.3). If possible, suppose that $z(t)$ is bounded. Without any loss of generality we may assume that $z(t) > 0$ for $t \geq t_0 \geq \sigma$. Let t_1 and $t_2 (t_0 < t_1 < t_2)$ be consecutive zeros of $y(t)$. So (Hanan

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

1961) there exists a constant μ such that $y(t) - \mu z(t)$ has a double zero at $t_3 \in (t_1, t_2)$. S $x(t) = y(t) - \mu z(t)$ is a solution of (3.3), then $x''(t_3) = 0$ implies that $x(t) \equiv 0$ and hence and $z(t)$ are linearly dependent, a contradiction. So $x''(t_3) \neq 0$. We assume that $x''(t_3)$ The case $x''(t_3) < 0$ may be treated similarly. Thus, there exists a $\delta > 0$ such that $x''(t)$ for $t \in [t_3, t_3 + \delta)$. We claim that $x''(t) > 0$ for $t \geq t_3$. If not, there exists a $t_4 > t_3$ such $x''(t_4) = 0$ and $x(t) > 0$, $x'(t) > 0$ and $x''(t) > 0$ for $t \in (t_3, t_4)$. Now integrating (3.3) fro to t_4 we get

$$0 > -x''(t_3) = \int_{t_3}^{t_4} Q(t)x(t)\, dt > 0,$$

a contradiction. Hence $x(t) > 0, x'(t) > 0$ and $x''(t) > 0$, for $t > t_3$. Consequently, $x(t)$ as $t \to \infty$. Since $z(t)$ is bounded, then $y(t) \to \infty$ as $t \to \infty$, a contradiction to the oscill nature of $y(t)$. Thus, $z(t)$ is unbounded.

Hence the lemma is proved.

If $g(t) = t - \tau(t)$, where $\tau \in C([\sigma, \infty), R)$ is such that $\tau(t) > 0$ and is bounded, (3.2) with $r(t) \equiv 1$ and $\gamma = 1$ takes the form

$$y'''(t) + p(t)y(t - \tau(t)) = 0.$$

THEOREM 3.9. *Suppose that* $p(t) \leq 0$ *and* $\lim_{t\to\infty} \inf t^3|p(t)| > \dfrac{2}{3\sqrt{3}}$.

Then bounded solutions of (3.4) *are oscillatory.*

Proof. Suppose that $y(t)$ is a bounded nonoscillatory solution of (3.4). Wit any loss of generality we may assume that $y(t) > 0$ and $y(t - \tau(t)) > 0$ for $t \geq t_0 \geq \sigma$. S $y'''(t) = -p(t)y(t - \tau(t)) \geq 0$ for $t \geq t_0$, then $y'(t)$ and $y''(t)$ are nonoscillatory. If $y''((t)$ for large t, then $y'(t) \to \infty$ as $t \to \infty$ and hence $y(t) \to \infty$ as $t \to \infty$, a contradiction. T $y''(t) < 0$ for $t \geq t_1 \geq t_0$.

If $y'(t) < 0$ for large t, then $y(t) \to -\infty$ as $t \to \infty$, a contradiction. So, $y'(t) >$ $t \geq t_2 \geq t_1$. Clearly, $y(t)$ is a bounded nonoscillatory solution of

$$z''' + p(t)\frac{y(t - \tau(t))}{y(t)}z = 0, \quad t \geq t_2.$$

From Lemma 3.6 we have $\lim_{t\to\infty} \dfrac{y(t - \tau(t))}{y(t)} = 1$. Consequently, Lemma 3.7 implies

$$\lim_{t\to\infty} \inf t^3 \left| p(t)\frac{y(t - \tau(t))}{y(t)} \right| = \lim_{t\to\infty} \inf t^3|p(t)| > \frac{2}{3\sqrt{3}}.$$

From Theorem 5.7 due to Hanan (1961), it follows that (3.5) admits an oscill solution. So, by Lemma 3.8, $y(t)$ is unbounded. This contradiction completes the proof o theorem.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

EXAMPLE. All bounded solutions of $y'''(t) - y\left(t - \frac{\pi}{2}\right) = 0, \quad t \geq 0$ are oscillatory. In particular, $y(t) = \sin t$ is an oscillatory solution of the equation.

4. Equation (1.4) is considered in this section with $\sigma = 0$ and oscillatory $f(t)$. Sufficient conditions have been obtained for all solutions/all bounded solutions of (1.4) to be oscillatory.

THEOREM 4.1. *Suppose that* $p(t) \geq 0$, $\int_0^\infty \frac{dt}{r(t)} = \infty$ *and there exists of function* $F \in C^2([0,\infty), R)$ *such that* $rF'' \in C^1([0,\infty), R)$, $(r(t)F''(t))' = f(t)$ *and* $\int_0^\infty p(t)\left(F_\pm(g(t))\right)^\gamma dt = \infty$, *where* $F_\pm(t) = \max\{F(t), 0\}$ *and* $F_-(t) = \max\{-F(t), 0\}$.

Then all solutions of (1.4) *are oscillatory.*

Proof. Let $y(t)$ be a nonoscillatory solution of (1.4) on $[0,\infty)$. We assume that $y(t) > 0$ for large t. The case when $y(t) < 0$ for large t may be treated similarly. So there exists a $t_0 > 0$ such that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_0$. Setting $x(t) = y(t) - F(t)$ for $t \geq t_0$, we obtain

$$(r(t)x''(t))' = -p(t)y^\gamma(g(t)) \leq 0. \tag{4.i}$$

So $x''(t) > 0$ or < 0 for large t. Thus, $x(t) > 0$ or < 0 for large t. Clearly, $x(t) < 0$ for $t \geq t_1 \geq t_0$ implies that $0 < y(g(t)) < F(g(t))$ for $t \geq t_2 \geq t_1$. Consequently,

$$\int_0^\infty p(t)(F_-(g(t)))^\gamma \, dt = \int_0^{t_2} p(t)(F_-(g(t)))^\gamma \, dt < \infty,$$

a contradiction. So $x(t) > 0$ for $t \geq t_1$.

If $x''(t) < 0$ for $t \geq t_3 \geq t_1$, then from (4.1) we obtain

$$x'(t) \leq x'(t_3) + r(t_3)x''(t_3) \int_{t_3}^t \frac{ds}{r(s)} < 0$$

for large t. Thus, $x(t) < 0$ for large t, a contradiction. Hence $x''(t) > 0$ for $t \geq t_3$. Integrating (4.1) from t_3 to t, we get $\int_{t_3}^t p(s)y^\gamma(g(s))\, ds < r(t_3)x''(t_3)$, that is, $\int_{t_3}^\infty p(t)y^\gamma(g(t))\, dt < \infty$.

Clearly, $x(t) > 0$ for $t \geq t_3$ implies that $F(g(t)) < y(g(t))$ for $t \geq t_4 \geq t_3$. Thus,

$$\int_{t_4}^\infty p(t)(F_+(g(t)))^\gamma \, dt < \int_{t_4}^\infty p(t)y^\gamma(g(t))\, dt < \infty,$$

that is, $\int_0^\infty p(t)(F_+(g(t)))^\gamma \, dt < \infty$, a contradiction, which completes the proof of the theorem.

The following example illustrates this theorem.

EXAMPLE. Consider $y'''(t) + y\left(t - \frac{\pi}{2}\right) = -2\cos t, t \geq 0$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Here $F(t) = 2\sin t$,

$$\int_0^\infty F_+\left(t - \frac{\pi}{2}\right) dt = -2\sum_{n=1}^{\infty} \int_{(4n-3)\frac{\pi}{2}}^{(4n-1)\frac{\pi}{2}} \cos t \, dt = 4\sum_{n=1}^{\infty} \cos 2n\pi = \infty$$

and $$\int_0^\infty F_-\left(t - \frac{\pi}{2}\right) dt = 2 + 2\sum_{n=1}^{\infty} \int_{(4n-1)\frac{\pi}{2}}^{(4n+1)\frac{\pi}{2}} \cos t \, dt = 2 + 4\sum_{n=1}^{\infty} \cos 2n\pi = \infty.$$

So all solutions of the equation are oscillatory. In particular, $y(t) = \sin t$ is an oscillato solution of the equation.

THEOREM 4.2. *Suppose that* $p(t) \le 0, \int_0^\infty \frac{dt}{r(t)} = \infty$ *and there exists a bound function* $F \in C^2([0,\infty), R)$ *such that* $rF'' \in C^1([0,\infty), R), (r(t)F''(t))' = f(t)$ *and*

$$\int_0^\infty p(t)\left(F_\pm(g(t))\right)^\gamma dt = -\infty,$$

where F_+ *and* F_- *are same as in Theorem* 4.1. *Then all bounded solutions of* (1.4) *are oscil tory.*

The proof of this theorem is similar to that of Theorem 4.1 and hence is omitted.

THEOREM 4.3. *Suppose that* $p(t) \le 0, \int_0^\infty \frac{dt}{r(t)} = \infty$,

$$\limsup_{t\to\infty} \left[\frac{1}{t}\int_0^t ds \int_0^s \frac{d\theta}{r(\theta)} \int_0^\theta f(t)du\right] = +\infty$$

and $$\liminf_{t\to\infty} \left[\frac{1}{t}\int_0^t ds \int_0^s \frac{d\theta}{r(\theta)} \int_0^\theta f(u)du\right] = -\infty.$$

Then all bounded solutions of (1.4) *are oscillatory.*

Proof. Suppose that $y(t)$ is a bounded nonoscillatory solution of (1.4) such th $|y(t)| \le M$. There exists a $t_0 \ge 0$ such that $y(t) > 0$ and $y(g(t)) > 0$ for $t \ge t_0$. The case wh $y(t)$ is ultimately negative may similarly be dealt with. Setting $x(t) = y(t) - h(t)$, where

$$h(t) = \int_0^t ds \int_0^s \frac{d\theta}{r(\theta)} \int_0^\theta f(u)du,$$

we obtain $(r(t)x''(t))' + p(t)\left[x(g(t)) + h(g(t))\right]^\gamma = 0$. Hence $(r(t)x''(t))' \ge 0$ for large t. Cons quently, $x(t), x'(t)$ and $x''(t)$ are nonoscillatory. If $x(t) < 0$ for large t, then $h(t) > 0$ for large a contradiction. So $x(t) > 0$ for large t. If $x''(t) > 0$ for large t, then from the given conditio it follows that $x(t) \to \infty$ as $t \to \infty$. Now $x(t) \le M - h(t)$ implies that

$$\infty = \liminf_{t\to\infty} x(t) \le M - \limsup_{t\to\infty} h(t) = -\infty,$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

a contradiction. Hence $x''(t) < 0$ for large t. Consequently, $x'(t) > 0$ for large t. Let $x(t) > 0, x'(t) > 0$ and $x''(t) < 0$ for $t \geq t_1 \geq t_0$. Clearly,

$$x(t) = x(t_1) + (t - t_1)x'(t_1) + \frac{(t - t_1)^2}{2} x''(\alpha),$$

$t_1 < \alpha < t$ implies that $x(t) \leq tx'(t_1) + x(t_1) - t_1 x'(t_1) \leq Lt$ for large t. Where $L > 0$ is a constant. Hence $\lim_{t\to\infty} \sup x(t)/t \leq L$. On the other hand, $x(t) > -h(t)$ implies that

$$\lim_{t\to\infty} \sup \frac{1}{t} x(t) > -\lim_{t\to\infty} \inf \frac{1}{t} h(t) = \infty,$$

a contradiction. Hence $y(t)$ is oscillatory.

This completes the proof of the theorem.

EXAMPLE 1. Consider $y'''(t) - y\left(t - \frac{3\pi}{2}\right) = -2\cos t, t > 0.$

If $F(t) = 2\sin t$, then all the conditions of Theorem 4.2 are satisfied and $y(t) = \sin t$ is a bounded oscillatory solution of the equation. We may note that Theorem 4.3 cannot be applied to this example because in this case

$$\lim_{t\to\infty} \left[\frac{1}{t} \int_0^t du \int_0^s dx \int_0^x (-2\cos u)\, ds\right] = \lim_{t\to\infty} \frac{1}{t}[2\sin t - 2t] = -2.$$

EXAMPLE 2. The equation $y'''(t) - e^t y\left(t - \frac{\pi}{2}\right) = (1 + e^t)\cos t, t \geq 1$ admits a bounded oscillatory solution $y(t) = -\sin t$. Clearly, all the assumptions of Theorem 4.3 are satisfied. Indeed,

$$\frac{1}{t}\int_1^t ds \int_1^s d\theta \int_1^\theta (1 + e^u)\cos u\, du = A\,t + B + \frac{C}{t} - \frac{\sin t}{t} + \frac{1}{4t} e^t(\sin t - \cos t)$$
$$= B + \frac{C}{t} - \frac{\sin t}{t} + \frac{e^t}{4t}\left(\sin t - \cos t + \frac{4At^2}{e^t}\right),$$

where A, B and C are constants, implies that

$$\lim_{t\to\infty} \sup \left[\frac{1}{t}\int_0^t ds \int_0^s \frac{d\theta}{r(\theta)} \int_0^\theta f(u)du\right] = +\infty$$

and $$\lim_{t\to\infty} \inf \left[\frac{1}{t}\int_0^t ds \int_0^s \frac{d\theta}{r(\theta)} \int_0^\theta f(u)du\right] = -\infty.$$

EXAMPLE 3. Clearly, Theorem 4.3 holds for the equation $y'''(t) - \frac{1}{t^5} y(t - \tau) = (18 - t^2)\, t\cos t + (6 - 9t^2)\sin t, t \geq 1$ because $\frac{1}{t}\int_0^t ds \int_1^s d\theta \int_1^\theta f(u)\, du = t^2\left[\sin t + \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t^3}\right],$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

where A, B and C are constants. But Theorem 4.2 cannot be applied to the above equation because $F(t) = t^3 \sin t$ and hence $F_+(t-\tau) \leq (t-\tau)^3$ implies that $p(t)F_+(t-\tau) \geq -\frac{(t-\tau)^3}{t^5}$. Consequently,

$$\int_1^\infty p(t)F_+(t-\tau)\, dt > -\infty.$$

References

Erbe, L. (1976) : Oscillation and asymptotic behaviour of solutions of third order differential delay equations, *SIAM J. Math. Anal.* 7, 491.

Grace, S. R. and **Lalli, B. S.** (1985) : On oscillation and nonoscillation of general functional differential equations, *J. Math. Anal. Appl.* 109, 522.

Gregus, M. (1987) : *"Third Order Linear Differential Equations,"* D. Reidel Publishing Company, Boston, Tokyo.

Hanan, M. (1961) : Oscillation criteria for third-order linear differential equations, *Pacific J. Math.* 11, 919, MR 26, 2695.

Kartsatos, A. G. and **Manougian, M. N.** (1974) : Perturbations causing oscillations of functional-differential equations, *Proc. Amer. Math. Soc.* 43, 111.

Keener, M. S. (1971) : On the solutions of certain linear nonhomogeneous second order differential equations, *Applicable Analysis*, 1, 57.

Kusano, T. and **Onose, H.** (1973) : Nonlinear oscillation of a sublinear delay equation of arbitrary order, *Proc. Amer. Math. Soc.* 40, 219.

Kusano, T. and **Onose, H.** (1974) : Oscillations of functional differential equations with retarded argument, *J. Differential Equations*, 15, 269.

Ladas, G., Sficas, Y. G. and **Stavroulakis, I. P.** (1984) : Necessary and sufficient conditions for oscillations of higher order delay differential equations, *Trans. Amer. Math. Soc.* 285, 81.

Ladde, G. S., Lakshmikantham, V. and **Zhang, B. G.** (1987) : *"Oscillation Theory of Differential Equations with Deviating Arguments"*, Marcel Dekker, Inc. New York and Basel, 1987.

Parhi, N. and **Nayak, S. K.** (1984) : Nonoscillation of second order nonhomogeneous differential equations, *J. Math. Anal. Appl.* 102, 62, MR 85k : 34077, *Zentralblatt Band* 544, 34025.

Parhi, N. and **Das, P.** (1993) : On asymptotic property of solutions of linear homogeneous third order differential equations, *Bull. Un. Mat. Ital.* 7, 775.

Parhi, N. and **Das, P.** (1994) : Oscillatory and asymptotic behaviour of a class of nonlinear differential equations of third order, *Bull. Inst. Math. Acad. Senica* (to appear).

Staikos, V. A. and **Sficas, Y. G.** (1972) : Oscillatory and asymptotic behaviour of functional differential equations, *J. Differential Equations*, 12, 426.

DEPARTMENT OF MATHEMATICS
BERHAMPUR UNIVERSITY
BERHAMPUR 760 007
INDIA

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc., 86, 267–270 (1994)

THE PURE STATE SPACE OF CERTAIN C*-ALGEBRAS

ADEL M. ZAKI

(Received 18 November 1992; revision received 27 September 1993)

Abstract. In this work we consider the C^*-algebra introduced in (Archbold and Zaki 1992). The pure state space of this C^*-algebra is determined precisely.

1. Introduction. Let H be a Hilbert space. Let $\ell(H)$ be the C^*-algebra of all bounded linear operators on H. IF ξ is a unit vector in H, we shall denote by ω_ξ the state of $\ell(H)$ given by $\omega_\xi(x) =< x\xi, \xi >$ (for $x \in \ell(H)$).

Suppose that a C^*-algebra A acts non-degenerately on H. Then $\omega_\xi|_A$ is called a vector state of A.

By an ideal of A we shall always mean a norm-closed two sided ideal. The primitive ideals of A are precisely those ideals which are the kernel of some irreducible representations of A. Let Prim (A) denote the family of all primitive ideals given the Jacobson topology.

An ideal I of A is called primal if whenever $n \geq 1$ and $J_1, J_2, \ldots, J_n$ are ideals of A such that $J_1 J_2 \ldots J_n = (0)$, then $J_i \subseteq I$ for at least one value of i. (Archbold and Batty 1986) and (Archbold 1987) for further properties of primal ideals of C^*-algebras).

If S is a subset of the dual space A^*, we denote by $\bar{S}$ the closure of S in the weak* topology. The state space of A is denoted by $S(A)$ and the set of pure states by $P(A)$. Thus, the pure state space of A (Glimm 1960) is the space $\overline{P(A)}$.

2. Main Result. We start this section by introducing the following result which will be used throughout the proof of the main result.

LEMMA 2.0. *Let A be a C^*-algebra. Then every proper primal ideal of A is maximal if and only if, Prim (A) is Hausdorff.*

Proof. ($\Rightarrow$) if Prim (A) is not Hausdorff, there exists a net (P_α) in Prim (A) with distinct limits Q_1 and Q_2 in Prim (A). Now, using (Archbold and Batty 1986; Archbold and Zaki 1992 and Archbold 1987) we get $Q_1 \cap Q_2$ is primal but not maximal.

($\Leftarrow$) Let I be a proper primal ideal of A. We prove that I is maximal. Then the limit in (Archbold and Batty 1986, Archbold 1987 and Archbold and Zaki 1992) is unique. Hence, there exists a unique $Q \in$ Prim (A) with $Q \supseteq I$ which implies that $I = Q \in$ Prim (A) (Dixmier 1977 and Archbold and Zaki 1992).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Now suppose there exists an ideal J of A so that $J \supset I$ (where $\supset$ denotes the proper inclusion) with $J \neq A$. Again, using (Dixmier 1977 and Archbold and Zaki 1992), there exists $P \in$ Prim (A) and $P \supseteq J$. So $P \supseteq J \supset I$ and hence $P \supset I$, a contradiction.

Now, using the facts that maximal ideals are primitive and primitive ideals are primal, we get Prim (A) is Hausdorff $\Rightarrow$ Prim (A) = Primal (A).

Let $M_n(C)$ denote the C^*-algebra of all $n \times n$ complex matrices. Let B the C^*-subalgebra of $M_6(C)$ consisting of all matrices of the form $\begin{bmatrix} S & O \\ O & T \end{bmatrix}$ where $S \in M_4(G)$ and $T \in M_2(C)$. We define A to be the C^*-algebra of all sequences $x = (\chi_n), n \geq 1$, where $\chi_n = \begin{bmatrix} S_n(x) & 0 \\ 0 & T_n(x) \end{bmatrix} \in B$ which converges (in norm) to a matrix of the form $\begin{bmatrix} T(x) & 0 & 0 \\ 0 & T(X) & 0 \\ 0 & 0 & T(x) \end{bmatrix}$ where $T(x) \in M_2(C)$.

The algebraic operations in A are defined pointwise and the norm is the supremum norm.

For each $n \geq 1$, there are irreducible representations π_n and σ_n of A given by $\pi_n(x) = S_n(x)$ and $\sigma_n(x) = T_n(x)$ for $x \in A$. The only other irreducible representation is the representation σ_∞ given by $\sigma_\infty(x) = T(x)$ for $x \in A$.

Hence Prim (A) consists of the ideals $P_n, Q_n (n \geq 1)$ and Q_∞, where $P_n = \ker \pi_n$, $Q_n = \ker \sigma_n$ and $Q_\infty = \ker \sigma_\infty = \{x \in A : \chi_n \to 0$, as $n \to \infty)$.

Each P_n and each Q_n is an isolated point in Prim (A). However $P_n \to Q_\infty$ and $Q_n \to Q_\infty$, as $n \to \infty$ in the Jacobson topology (Dixmier 1977). In particular, Prim (A) is Hausdorff and so Primal (A) = Prim (A), where Primal (A) denotes the set of all proper primal ideals of A. Notice that A is unital, separable and liminal.

Let ψ be any state of $M_2(C)$. Let $x - (\chi_n)$ be in A and define ϕ such that $\phi = \psi.\sigma_\infty$.

Therefore, $\phi(x) = \psi(T(x))$.

It has been proved in (Archbold and Zaki 1992) that when ψ is the normalized trace defined on $M_2(C), \phi$ belongs to $\overline{P(A)}$. However, in general, using (Dixmier 1977), we have $\psi = \lambda\omega_\xi + (1-\lambda)\omega_\eta$ for some $\{\xi, \eta\}$ orthonormal basis for C^2 and $0 \leq \lambda \leq 1$. Therefore, $\phi(x) = (\lambda\omega_\xi + (1-\lambda)\omega_\eta)(T(x)) = \lambda < T(x)\xi, \xi > +(1-\lambda) < T(x)\eta, \eta >$.

Suppose $\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$ and $\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$ and notice that $\phi = w^* - \lim_{n\to\infty} \varphi_{\sqrt{\lambda}\gamma + \sqrt{1-\lambda}\,\delta} \circ \pi_n$, where γ, δ are given by $\gamma = \begin{bmatrix} \xi_1 \\ \xi_2 \\ O \\ O \end{bmatrix}$ and $\delta = \begin{bmatrix} O \\ O \\ \eta_1 \\ \eta_2 \end{bmatrix}$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Hence $\phi \in \overline{P(A)}$.

LEMMA 2.1. *Let ψ be any state of $M_4(C)$. Let $\phi = \psi \circ \pi_n$ for some $n \in N$. Then $\phi \in \overline{P(A)}$ iff ψ is pure.*

Proof. ($\leftarrow$) Let ψ be pure. So $\phi \in P(A)$.

($\rightarrow$) Let $\phi \in \overline{P(A)}$. Therefore,

$\phi_\alpha \rightarrow \phi$ for some $\phi_\alpha \in P(A)$.

We show that $\pi_{\phi_\alpha} \rightarrow \ker \pi_n$ in Prim (A).

Let U be an open neighborhood of $\ker \pi_n$. Then using (Dixmier 1977), $\cup = \{Q \in$ Prim $(A); Q \not\supseteq I\}$ for some ideal I of A. So $\pi_n(I) \neq (0)$

We show that $\ker \pi_{\phi_\alpha}$ belongs to U eventually.

Suppose not, then $\pi_{\phi_\alpha}(I) = (0)$ frequently and hence $\pi_\alpha(I) = (0)$ frequently. So we get $\phi(I) = \psi(\pi_n(I)) = 0$.

But $\pi_n(I)$ is a proper ideal of $M_4(C)$. Thus $\pi_n(I) = \{0\}$, a contradiction.

Since $\ker \pi_n$ is an isolated point of Prim (A), we get $\ker \pi_{\phi_\alpha} = \ker \pi_n$ eventually and hence $\phi_\alpha \in P(A/\ker \pi_n)$ eventually.

Define ψ_α in $S(M_4(C))$ so that, $\phi_\alpha = \psi_\alpha \circ \pi_n$ eventually.

Since ϕ_α is pure, then ψ_α is a pure state of $M_4(C)$. Moreover, $\psi_\alpha \circ \pi_n \rightarrow \psi \circ \pi_n$.

Hence $\psi_\alpha \rightarrow \psi$ and $\psi \in \overline{P(M_4(C))}$.

Thus, by considering $M_4(C)$ as $\ell e(C^4)$, ψ is a multiple of a pure state and hence pure, since ψ is a state.

The proof of the latter lemma could easily work to show the following:

LEMMA 2.2. *Let ψ be any state of $M_2(C)$: Let $\phi = \psi \circ \sigma_n$ for some $n \in N$. Then $\phi \in \overline{P(A)}$ iff ψ is pure.*

Now we are in a position to prove the main result of this work which determine precisely the pure state space of the C^*-algebra A.

PROPOSITION 2.3. $\overline{P(A)} = \{\psi \circ \sigma_\infty : \psi$ is any state of $M_2(C)\}; U\{\omega_\xi \circ \pi_n : n = 1, 2, \ldots$ and ξ is a unit vector in $C^4\}; U\{\omega_\eta \circ \pi_n : n = 1, 2, \ldots$ and η is a unit vector in $C^2\}$.

Proof. Firstly, we proved in the discussion preceding lemma 2.1 that $\psi \circ \sigma_\infty \in \overline{P(A)}$, where ψ is any state of $M_2(C)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Secondly, we prove that given ϕ in $\overline{P(A)}$, then ϕ must factor through some primitive ideal of A. Since $\phi \in \overline{P(A)}(\subseteq \overline{F(A)})$, then ϕ, must factor through a primal ideal of A (Archbold and Batty 1986). Since Prim (A) is Hausdorff,then the primal ideals are precisely the primitive ones. Finally, if ϕ factors through ker σ_∞, there is nothing to prove. If ϕ factors through ker π_n or ker σ_∞, for some n, then we can apply lemma 2.1 or lemma 2.2, respectively.

References

Archbold, R. J. and **Batty, C. J. K.** (1986) : "On factorial states of operator algebras. III", *J. Operator Theory, 15*, 53.

Archbold, R. J. and **Zaki, A. M.** (1992) : "Limits of pure states II", *J. Edin. Math. Soc., 35*, 227.

Archbold, R. J. (1987) : "Topologies for Primal ideals", *J. London Math. Soc., (2) 36*, 524.

Dixmier, J. (1977) : "*C^*-algebras*, North-Holland, Amesterdam.

Glimm, J. (1960) : "A Stone-Weierstrass theorems, for C^*-algebras" *Ann. Math., 72*, 216.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
ASSUIT UNIVERSITY
ASSUIT
EGYPT

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc., 86, 271–274 (1994)

FUZZY CONNECTEDNESS AND WEAKER FORMS OF FUZZY CONTINUITY

ABDULLA S. BIN SHAHNA

(*Received 12 April 1991; revision received 3 December 1991*)

1. Introduction. The purpose of this paper is to deal with the preservation of certain connectedness concepts under weaker forms of fuzzy continuity. It is seen that fuzzy connectedness is preserved not only under fuzzy continuous mappings, but is also preserved under some of its weaker forms. We end this paper by examining the preservation of fuzzy super connectedness and fuzzy strong connectedness introduced by Fatteh and Bassan (1985), under these weaker forms of fuzzy continuity.

Let X be a non-empty set. Fuzzy sets in X are denoted by λ, μ and ν etc.; 0 and 1 are fuzzy sets which map whole of X into 0 and 1, respectively. We emphasize that the definition of a fuzzy topological space (or fuzzy space) is used in the sense of Chang (1968) and will be denoted by the pair (X, τ) or simply by X. We shall not mention the terminologies and notations which are quite well-known by now. For those used in this paper and not mentioned, we refer to the paper (Azad (1981), Bin Shahna (1991), Fatteh and Bassan (1985) and Yalvac (1987)). However, we recall some definitions which will be used frequently in the sequel.

A fuzzy set λ in a fuzzy space X is called

(i) *fuzzy semiopen* (Azad 1981) if $\lambda \leq Cl$ Int λ.

(ii) *fuzzy regular open* (Azad 1981) if $\lambda =$ Int $Cl\lambda$.

(iii) *fuzzy α-open* (Bin Shahna 1991) if $\lambda \leq$ Int Cl Int λ.

(iv) *fuzzy preopen* (Bin Shahna 1991) if $\lambda \leq$ Int $Cl\lambda$.

Let $f : X \to Y$ be a mapping from a fuzzy space X to another fuzzy space Y. Then f is called

(i) *fuzzy semicontinuous* (Azad 1981) if the inverse image of each fuzzy open set in Y is a fuzzy semiopen set in X.

(ii) *fuzzy almost continuous* (Azad 1981) if the inverse image of each fuzzy regular open set in Y is a fuzzy open set in X; or equivalently $f^{-1}(\lambda) \leq$ (Int $Cl\lambda$) for each fuzzy open set λ of Y.

(iii) *fuzzy weakly continuous* (Azad 1981) if for each fuzzy open set λ in $Y, f^{-1}(\lambda) \leq$ Int$f^{-1}(Cl\lambda)$.

(iv) *fuzzy strongly semicontinuous* (Bin Shahna 1991) if the inverse image of each fuzzy open set in Y is a fuzzy α-open set in X.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(v) *fuzzy precontinuous* (Bin Shahna 1991) if the inverse image of each fuzzy open set in Y is a fuzzy preopen set in X.

2. Fuzzy Connected Spaces

DEFINITION 2.1. (Hutton 1975). *A fuzzy space X is called fuzzy connected if it has no proper fuzzy clopen set.*

DEFINITION 2.2. (Ali and Srivastava 1988). *The unit interval $I = [0,1]$ together with the fuzzy topology $\tau = \{0, \lambda, \lambda', \lambda \wedge \lambda', \lambda \vee \lambda', 1\}$ where $\lambda : I \to I$ is defined by $\lambda(t) = t$, for each $t \in I$, will be called the fuzzy 2-space and we shall denote it by 2_f.*

Below, we state some results (Theorems 2.3 to 2.6) without proofs. Similar results concerning various concepts for fuzzy connectedness can be found in (Ali and Srivastava 1988).

THEOREM 2.3. *A fuzzy space X is fuzzy connected if and only there is no proper fuzzy continuous mapping f (i.e., $f \neq 0, f \neq 1$) from X to 2_f.*

THEOREM 2.4. *Fuzzy continuous image of a fuzzy connected space is fuzzy connected.*

THEOREM 2.5. *Fuzzy almost continuous image of a fuzzy connected space is fuzzy connected.*

THEOREM 2.6. *Fuzzy weakly continuous image of a fuzzy connected space is fuzzy connected.*

THEOREM 2.7. *Fuzzy strongly semicontinuous image of a fuzzy connected space is fuzzy connected.*

Proof. Let $f : X \to Y$ be a fuzzy strongly semicontinuous mapping from a fuzzy connected space X onto a fuzzy space Y. Assume μ to be a proper fuzzy clopen set in Y. Since f is fuzzy strongly semicontinuous, $f^{-1}(\mu)$ is a fuzzy α-open as well as a fuzzy α-closed set in X. That is $f^{-1}(\mu)$ is a fuzzy semiopen, fuzzy preopen, fuzzy semiclosed and fuzzy preclosed set (Bin Shahna 1991, Theorem 1.6). Obviously $f^{-1}(\mu)$ is a proper fuzzy clopen set in X; a contradiction to the fact that X is fuzzy connected.

REMARK 2.8. The following example shows that fuzzy connectedness is preserved neither by fuzzy semicontinuous mappings nor by fuzzy precontinuous mappings.

Let λ, μ and ν be fuzzy sets in $I = [0,1]$ defined by

$$\lambda(x) = \begin{cases} 0, & 0 \leq x \leq 1/2, \\ 2x - 1, & 1/2 \leq x \leq 1; \end{cases}$$

$$\mu(x) = \begin{cases} 1 - 2x, & 0 \leq x \leq 1/2, \\ 0, & 1/2 \leq x \leq 1; \end{cases}$$

$$\text{and} \quad \nu(x) = \begin{cases} 0, & 0 \leq x \leq 1/2, \\ 1, & 1/2 < x \leq 1. \end{cases}$$

Clearly $\tau_1 = \{0, \lambda, \mu, \lambda \vee \mu, 1\}, \tau_2 = \{0, \nu, \nu', 1\}$ and $\tau_3 = \{0, \lambda', 1\}$ are fuzzy topologies on I. Moreover, (I, τ_1) and (I, τ_3) are fuzzy connected spaces but (I, τ_2) is not. Consider the mapping $f : I \to I$ defined by $f(x) = x$, for each $x \in I$.

Since ν and ν' are fuzzy semiopen sets in (I, τ_1), therefore, the mapping $f : (I, \tau_1) \to (I, \tau_2)$ is fuzzy semicontinuous. Also ν and ν' are fuzzy preopen sets in (I, τ_3), hence $f : (I, \tau_3) \to (I, \tau_2)$ is a fuzzy precontinuous mapping.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

3. Fuzzy Super and Fuzzy Strongly Connected Spaces

DEFINITION 3.1. (Fatteh and Bassan 1985). *A fuzzy space X is called fuzzy super connected if it has no proper fuzzy regular open set.*

DEFINITION 3.2. (Fatteh and Bassan 1985). *A fuzzy space X is called fuzzy strongly connected if it has no non-zero fuzzy closed sets λ and μ such that $\lambda + \mu \leq 1$.*

It has been shown that a fuzzy continuous image of a fuzzy super (resp. fuzzy strongly) connected space is fuzzy super (resp. fuzzy strongly) connected (Fatteh and Bassan 1985, Theorem 6.5. and Theorem 8.4.)

THEOREM 3.3. *A fuzzy almost continuous image of a fuzzy super connected space is fuzzy super connected.*

Proof. Let $f : X \to Y$ be a fuzzy almost continuous mapping from a fuzzy super connected space X onto a fuzzy space Y. Suppose Y is not fuzzy super connected. Then there exists a proper fuzzy regular open set λ. Hence λ is a non-zero fuzzy open set in Y and therefore $Cl\lambda \neq 1$ (Fatteh and Bassan; Theorem 6.1). Since f is fuzzy almost continuous, f is also fuzzy weakly continuous and hence $Clf^{-1}(\lambda) \leq f^{-1}(Cl\lambda)$. Because $\lambda \neq 0$ and $Cl\lambda \neq 1$, there exist y_1 and y_2 in Y such that $\lambda(y_1) \neq 0$ and $Cl(\lambda)(y_2) \neq 1$. Now f is onto, there exist x_1 and x_2 in X such that $f(x_1) = y_1$ and $f(x_2) = y_2$, so $f^{-1}(\lambda)(x_1) = \lambda f(x_1) = \lambda(y_1) \neq 0$.

Similarly $f^{-1}(Cl\lambda) \neq 1$. Thus $f^{-1}(\lambda)$ is a non-zero fuzzy open set in X such that $Clf^{-1}(\lambda) \neq 1$; this is a contradiction to the fact that X is fuzzy super connected.

COROLLARY 3.4. *Let $f : X \to Y$ be a subjective fuzzy W, almost open and fuzzy weakly continuous mapping. If X is fuzzy super connected, then so is Y.*

REMARK 3.4. The following example shows that fuzzy strong connectedness is not preserved by any one of the following weaker forms of fuzzy continuity : fuzzy almost continuity, fuzzy weakly continuity, fuzzy strong semicontinuity, fuzzy semicontinuity and fuzzy precontinuity.

Let μ_1, μ_2, μ_3 and μ_4 be fuzzy sets in I defined by

$$\mu(x) = \begin{cases} 3/4 & \text{if } x = 1/4, \\ 1, & \text{otherwise;} \end{cases}$$

$$\mu_2(x) = \begin{cases} 2/3, & \text{if } x = 1/4, \\ 1, & \text{otherwise;} \end{cases}$$

$$\mu_3(x) = \begin{cases} 1/4, & \text{if } x = 1/4, \\ 0, & \text{otherwise;} \end{cases}$$

$$\mu_4(x) = \begin{cases} 1, & \text{if } x = 3/4, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $\tau_1 = \{0, \mu_3, 1\}, \tau_2 = \{0, \mu_1, \mu_2, 1\}$ and $\tau_3 = \{0, \mu_4, 1\}$ are fuzzy topologies on I. Also (I, τ_1) and (I, τ_3) are fuzzy strongly connected spaces but (I, τ_2) is not. Define a mapping $f : I \to I$ by $f(x) = x$, for each $x \in I$.

Now $f : (I, \tau_1) \to (I, \tau_2)$ is a fuzzy almost continuous mapping and hence it is fuzzy weakly continuous (Azad 1981, Remark 8.2) and $f : (I, \tau_3) \to (I, \tau_2)$ is a fuzzy strongly semi-

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

continuous mapping; hence it is fuzzy semicontinuous and fuzzy precontinuous (Bin Shahna 1991, Theorem 3.6).

REMARK 3.6. The following example shows that fuzzy supper connectedness is preserved neither by fuzzy weakly continuous mappings nor by fuzzy precontinuous mappings.

Let λ_1 and λ_2 be fuzzy sets in I defined by

$$\lambda_1(x) = \begin{cases} 0, & 0 \le x \le 1/2, \\ 2x-1, & 1/2 \le x \le 1; \end{cases}$$

$$\text{and} \quad \lambda_2(x) = \begin{cases} 1, & 0 \le x \le 1/4, \\ 2-4x, & 1/4 \le x \le 1/2, \\ 0, & 1/2 \le x \le 1. \end{cases}$$

Clearly $\tau_1 = \{0, \lambda_1', \lambda_2', \lambda_1' \wedge \lambda_2', 1\}, \tau_2 = \{0, \lambda_1 \vee \lambda_2, 1\}$ and $\tau_3 = \{0, \lambda_1, \lambda_2, \lambda_1 \vee \lambda_2, 1\}$ are fuzzy topologies on I. Moreover (I, τ_1) and (I, τ_2) are fuzzy super connected spaces but (I, τ_3) is not because λ_1 and λ_2 are proper fuzzy regular open sets. Consider the mapping $f : I \to I$ defined by $f(x) = x$, for each $x \in I$.

Now $f : (I, \tau_1) \to (I, \tau_3)$ is a fuzzy weakly continuous mapping and $f : (I, \tau_2) \to (I, \tau_3)$ is a fuzzy precontinuous mapping.

References

Ali, D. M. and **Srivastava, A. K.** (1988) : On fuzzy connectedness, *Fuzzy sets and Systems, 28,* 203.

Azad, K. K. (1981) : On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity, *J. Math. Anal. Appl. 82,* 14.

Bin Shahna, A. S. (1991) : On fuzzy strong semicontinuity and fuzzy precontinuity, *Fuzzy Sets and Systems, 44,* 303.

Chang, C. L. (1968) : Fuzzy topological spaces, *J. Math. Anal. Appl., 24,* 182.

Fatteh, U. V. and **Bassan, D. S.** (1985) : Fuzzy connectedness and its stronger forms, *J. Math. Anal. Appl., 111,* 449.

Hutton, B. (1975) : Normality in fuzzy topological spaces, *J. Math. Anal. Appl., 50,* 74.

Hutton, B. (1980) : Product of fuzzy topological spaces, *Topology Appl., 11,* 59.

Long, P. E. and **Carnahan, D. A.** (1973) : Comparing almost continuous functions, *Proc. Amer. Math. Soc., 38,* 413.

Noiri, T. (1974) : On weakly continuous mappings, *Proc. Amer. Math . Soc.* 46, 120.

Yalvac, T. H. (1987) : Fuzzy sets and functions on fuzzy spaces, *J. Math. Anal. Appl., 126,* 409.

Zadeh, L. A. (1965) : Fuzzy sets, *Inform and Control, 8,* 338.

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF ALLAHABAD
UTTAR PRADESH
INDIA

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

l. Cal. Math. Soc., 86, 275–278 (1994)

ONOSCILLATION FOR A CLASS OF FORCED SECOND ORDER NON-LINEAR DIFFERENTIAL EQUATIONS WITH DAMPING

K. V. V. SESHAGIRI RAO

(*Received 7 June 1991; revision received 28 September 1992*)

Abstract. Sufficient conditions for the solutions of $(rx')'+p(t)g(x')+q(t)h(x) = f(t)$ be nonscillatory are presented.

1. Introduction. Consider the second order nonlinear differential equation,

$$(rx')' + p(t)g(x') + q(t)h(x) = f(t) \tag{1.1}$$

which $r > 0, p, q, f$ are continuous functions on $[0, \infty)$, and g, h are continuous real valued nctions of R.

One of the prototypes of this class of equations is

$$(rx')' + q(t)x^{\alpha} = 0, \tag{1.2}$$

> 0 the quotient of odd integers. When $r(t) = 1$, and $q(t) > 0$, this equation is subjected numerous investigations (cf. Bellman (1968), Hille (1970), P. K. Wong (1963, 1970)). An tensive analysis of the asymptotic behaviour of solutions of the equation (1.2) may be found aliaferro 1978). For nonoscillatory behaviour of the solutions of the equation (1.1) with $t) = 0$ and $q(t) < 0$, we refer the reader to (Erbe, Rao and Rao 1984), and for oscillatory haviour we refer to (Rao 1991). Our interest here is to investigate properties of the solutions the equation (1.1) under the influence of a forcing function $f(t)$ and damping term. We nsider the solutions that exist on $[a, \infty), a > 0$.

We recall that a solution of (1.1) or (1.2) is said to be nonoscillatory on its maximal erval of existence $[0, b), 0 < b \leq +\infty$, in case there exists $a_1 \geq 0$ such that $x(t) \neq 0$ on $[a_1, b)$. he solution $x(t)$ which exists on $[0, +\infty)$ is said to be oscillatory on $[0, +\infty)$, if for each $a_1 \geq 0$ ere exists a_2, a_3 with $a_1 < a_2 < a_3$ and $x(a_2) < 0, x(a_3) > 0$, and is said to be of Z-type if it s arbitrarily large zeros, but is ultimately non-negative or non-positive.

In Section 2, we obtain the nonoscillation criteria for the equation (1.1) when the mping is linear, and we study the effect of nonlinear damping in Section 3.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Section 2. We need the following hypotheses at different stages

$$
\begin{aligned}
&p \in C'[a,\infty) \text{ for } a \geq 0\\
&p(t) \leq 0,\ q(t) \geq 0 \text{ for all } t \in [0,\infty)\\
&f(t) \geq 0 \text{ for } t \geq a \geq 0\\
&p'(t) - q(t) \geq 0 \text{ for large } t\\
&xh(x) > 0 \text{ for } x \neq 0\\
&ug(u) > 0 \text{ for } u \neq 0.
\end{aligned}
$$

Firstly, we obtain nonoscillation criteria when $g(u) = u$, and $h(x) = x$.

THEOREM 2.1. *Suppose Hypotheses H_1, H_2, H_3, H_4 hold for large t. Then all solutions of* (1.1) *are nonoscillatory.*

Proof. If possible let $x(t)$ be an oscillatory or non-negative Z-type solution of the equation (1.1). Then for large t, there exists a, b such that $x(a) = 0, x(b) = 0, x'(a) \geq 0, x'(b) \leq 0$ on (a, b), then from (1.1),we have,

$$
\begin{aligned}
0 > r(b)x'(b) - r(a)x'(a) &= \int_a^b f(t)dt - \int_a^b p(t)x'(t)dt - \int_a^b q(t)x(t)dt\\
&= \int_a^b f(t)dt + p(a)x(a) - p(b)x(b) + \int_a^b [p'(t) - q(t)]x(t)dt > 0,
\end{aligned}
$$

a contradiction.

Now let $x(t)$ be a non-positive Z-type solution of (1.1), if possible. Then there exists a, b for large t such that $x(a) = 0 = x(b), x'(a) = 0, x'(b) = 0$, and $x(t) < 0$ on (a, b).

Also there exists $c \in (a, b)$ such that $x'(c) = 0$ such that $x'(t) > 0$ on (c, b). Integrating (1.1) between c and b we obtain,

$$
\begin{aligned}
0 = r(b)x'(b) - r(c)x'(c) &= \int_c^b f(t)dt - \int_c^b p(t)x'(t)dt - \int_c^b q(t)x(t)dt\\
&> 0, \quad \text{a contradiction.}
\end{aligned}
$$

This completes the proof of theorem.

If $g(u) = u$, and $h(x)$ satisfies H_5, we have the following result.

THEOREM 2.2. *Suppose H_1, H_2, H_3, H_4, H_5 hold. Then all solutions $x(t)$ of* (1.1) *which satisfy the inequality $z - h(z) \geq 0$ in any interval where $x(t) > 0$ are non-oscillatory.*

Proof. As in Theorem 2.1. let $x(t)$ be an oscillatory or non-negative Z-type solution if possible. Then there exists a, b for large t such that $x(a) = 0, x(b) = 0, x'(a) \geq 0$, $x'(b) \leq 0$ on (a, b). An integration of the equation (1.1) between a and b yields

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$
\begin{aligned}
0 > r(b)x'(b) - r(a)x'(a) &= \int_a^b f(t)dt - \int_a^b p(t)x'(t)dt - \int_a^b q(t)h(x(t))dt \\
&= \int_a^b f(t)dt + p(a)x(a) - p(b)x(b) + \int_a^b [p'(t)x(t) - q(t)h(x(t)]dt \\
&= \int_a^b f(t)dt + \int_a^b [p'(t) - q(t)\{h(x(t))/x(t)\}]x(t)dt \\
&\geq \int_a^b f(t)dt + \int_a^b [p'(t) - q(t)]x(t)dt > 0,
\end{aligned}
$$

which is a contradiction. The second part of the proof is similar to that in the proof of Theorem 2.1., and this completes the proof.

Section 3. In this section we establish nonoscillation criteria for the solutions of (1.1) when g is nonlinear.

THEOREM 3.1. *Suppose H_2, H_3, H_5, and H_6 hold. Also suppose that $\lim_{t\to\infty}[f(t)/q(t)] = \infty$. Then all the bounded solutions of* (1.1) *are nonoscillatory.*

Proof. Let $x(t)$ be a bounded solution of the equation (1.1) which exists on $[t_1, \infty)$ for some $t_1 > 0$ such that $|x(t)| < K$ for $t \geq t_1$. From the hypothesis, there exist a $t_2 \geq t_1$ such that $f(t) > h(K)q(t)$ for $t \geq t_2$.

If possible let $x(t)$ be oscillatory or non negative Z-type solution. Then there exist a and b such that $b > a > t_2$ with $x(a) = 0, x(b) = 0, x'(a) \geq 0, x'(b) \leq 0$, and $x(t) > 0$ on (a, b). Also there exists $c \in (a, b)$ such that $x'(c) = 0$ and $x'(t) > 0$ on (a, c) and $x'(t) < 0$ on (c, b). Integrating (1.1) between a and c, we have

$$
\begin{aligned}
0 = r(c)x'(c) - r(a)x'(a) &= \int_a^c f(t)dt - \int_a^c p(t)g(x'(t))dt - \int_a^c q(t)h(x(t))dt \\
&\geq \int_a^c [f(t) - q(t)h(K)]dt > 0,
\end{aligned}
$$

which is a contradiction. Now, if possible, let $x(t)$ be a nonpositive Z-type solution with double zeros $d_1, d_2 \geq t_2$ such that $x(d_1) = x'(d_1) = 0, x(d_2) = x'(d_2) = 0$ and $x(t) < 0$ on (d_1, d_2). Also there exists $d_3 \in (d_1, d_2)$ such that $x'(d_3) = 0, x'(t) < 0$ on (d_1, d_3) and $x'(t) > 0$ on (d_3, d_2). Integrating the equation (1.1) between d_3 and d_2, we have

$$
\begin{aligned}
0 = r(d_2)x'(d_2) - r(d_3)x'(d_3) &= \int_{d_3}^{d_2} [f(t) - p(t)g(x'(t)) - q(t)h(x(t))]dt \\
&> 0,
\end{aligned}
$$

which is again a contradiction. This completes the proof of the theorem.

We change the condition on $f(t)$ and $q(t)$ while restricting $h(x)$, and obtain the following result.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

THEOREM 3.2. *Suppose H_4, H_5 hold. If $f(t) \geq h(K)q(t)$ for large t, where K then all solutions of* (1.1) *which satisfy the inequality $z \leq K$ in any interval where $x(t) >$ nonoscillatory.*

The proof follows on similar lines of the proof of Theorem 3.1. Hence the detail omitted.

The following examples illustrate the results obtained.

EXAMPLE 3.3. Consider the differential equation

$$\left(\frac{1}{t^7}x'\right)' - \frac{1}{t^2}(x')^3 + \frac{1}{t^{58/5}}x^{1/5} = \frac{28}{t''} + \frac{1}{t_t^{12}}.$$

Clearly all the conditions of Theorem 3.1 are satisfied. Hence all bounded solutio this equation are nonoscillatory. Indeed $x(t) = 1/t^2$ is one such solution.

EXAMPLE 3.4. Consider the equation

$$\left(\frac{x'}{(t+1)}\right)' + (t+1)(x')^3 - \frac{x^{1/3}}{(t+1)^{20/3}} = \frac{3}{(t+1)^4} - \frac{1}{(t+1)^3} + \frac{1}{(t+1)^7}.$$

The conditions of Theorem 3.2 are satisfied with $K = 1$. Therefore, we note that solutions of this equation of the type specified in Theorem 3.2 are nonoscillatory. Cl $x(t) = 1/(1+t)$ is such a solution.

References

Bellman, R. (1968) : *Stability theory of differential equations,* Academic Press, New York.

Erbe, L., Sreehari Rao, V. and **Seshagiri Rao, K. V. V.** (1984) : Nonoscillation and asymptotic behaviour of certain cl second order nonlinear differential equations. *Proc. Camb. Phil. Soc., 95,* 155.

Hillé, E. (1970) : Aspects of Emden's equation, *J. Pac. Sci., Univ. Tokyo Sect. IA Math., 17,* 11.

Hille, E. (1970) : Some aspects of Thomas-Fermi equation, J. Anal. Math. 23, 147.

Seshagiri Rao, K. V. V. (1991) : Oscillating periodic solutions of second order nonlinear differential equations (Prepri

Taliaferro, S. D. (1978) : Asymptotic behaviour of solutions of $y'' = \phi(t)y^\lambda$ *J. Math. Anal. Appl., 66,* 95.

Wong, P. K. (1963) : Existence and asymptotic behaviour of proper solutions of a class of second order nonlinear diffe equations, *Pacific J. Math., 13,* 737.

Wong, P. K. (1970) · Bounds for solutions to a class of nonlinear second order differential equations, *J. Diff. Eqns., 7,*

DEPARTMENT OF MATHEMATICS
KAKATIYA INSTITUTE OF TECHNOLOGY AND SCIENCE
WARANGAL 506015 (A.P.)
INDIA

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc., 86, 279–290 (1994)

ON COMPUTATION OF TRIANGULATION WITH CONTINUOUS REFINEMENT OF GRID SIZE

SHABNAM KHAN, D. P. SHUKLA AND N. P. S. BAWA

(Received 6 December 1991; revision received 21 June 1992)

Abstract. This paper deals with the computation of triangulation with continuous refinement of grid size. The triangulation considers are J_3 and K_3 which are both regular triangulation of $]0,1] \times R^n$ for $n = 3$. The simplices of these triangulation have their vertices on straight line segments. We give a simple description of these two triangulation with $]0,1] \times R^n$ with continuous refinement of grid size.

1. Introduction. There are number of triangulations developed by different workers for R^n and S^n and $]0,1] \times R^n$ in order to compute the fixed point of a continuous function. Just as, K_1 triangulation of R^n is due to Kuhn (1960) as well as of Freudenthal (1942). Similarly the triangulation J_1 of R^n is due to Tucker given in the Lemke (1970) but it is also found in Whitney (1957). Again the triangulation obtained is S^n are $K_2(m)$ and $J_2(m)$. With the help of these triangulations Brouwer's fixed point theorem and Kakutani's fixed point theorem have been proved and many other conditions related to them.

But the algorithm based on these triangulation is not as efficient as compared to the triangulation due to continuous refinement of grid size developed by Eaves (1972), Eaves and Saigal (1972). These triangulations are the generalization of K_1-based and J_1-based simplices resulting in K_3 and J_3 triangulation.

Vector labelling has been used for the labelling of vertices of triangulation and particular starting point is called pivot point. The advantage of the triangulation due to continuous refinement of grid size is that it provides improved computational technique for fixed point algorithm to calculate the fixed point of the mapping to greater accuracy.

2. Notations and Definitions. As usual R denotes the real line, Z the set of integer, π_n and π_{n+1} denote the group of permutation of the sets $N = \{1, \ldots, n\}$ and $N_o = \{0, 1, \ldots, n\}$ respectively used to label the co-ordinates of n and $(n+1)$-dimensional space, u^i denotes the ith unit vector in R^n for $i \in \pi_n, v^j$ denotes jth unit vector in $R^{n+1}, j \in \pi_{n+1}$.

The standard n-dimensional closed simplex S^n is the convex hull of v^o through v^n, and K-dim. Simplex (or K-simplex) σ is the relative interior of the convex hull of $(K+1)$ affinely independent points $y^o, \ldots, y^k$ called its vertices. We denote the simplex $\sigma =< y^o, \ldots, y^k)$. A simplex τ is a face of σ if its vertices are subset of vertices of σ and the closure of $(K-1)$-dimensional face of K-simplex σ is called a facets of σ. Two simplices are said to be adjacent if they have a common facet.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

A triangulation G of S^n is a finite collection of open n-simplices such that an o n-simplices, together with all their open faces, form a partition of S^n. $s \in R^n$ denote sign vector. The mesh of triangulation G is $\sup_{\sigma \in G}$ diam σ. U.S.C. denotes the upper s continuous and A^{n*} is the set of collection of subset of a set A.

2.1. Since the process of triangulation in R^n and S^n constructed by various approa is used for the computation of approximate fixed point of a function, so that refinement in triangulation provides a better algorithm in the determination of the approximate fixed p with greater accuracy.

As the triangulation of continuous refinement of grid size based on the work of E (1972), Eaves and Saigal (1972) mostly required the concept of homotopy and vector label so we will discuss them to develop algorithm applied for the determination of fixed point of function.

DEFINITION. 2.2. *Let $f^i : R^n \to R^n$ be a continuous function for $i = 0, 1$.*

Then $h : [0,1] \times R^n \to R^n$ is a homotopy from f^o to f^1, if h is continuous $h(i, X^T)^T = f^i(X) \forall X \in R^n$ and $i = 0, 1$.

We will not consider the triangulation required to construct the homotopy.

DEFINITION. 2.3. *Let $\delta k \to 0$ then the triangulation G of $]0,1] \times R^n$ is said to continuous refinement of grid size if G satisfies the following:*

(i) $y \in G^o \Longrightarrow y_o = 2^{-k}$ for $k = 0, 1, 2, \ldots$;

(ii) $G_k = \{\sigma \in G^n | \sigma \subseteq R^n(k)$ *is the simplex of* $R^n(k)\}$ *for* $k = 0, 1, 2, \ldots$;

(iii) $\sigma \subseteq]0, 2^{-k}] \times R^n \Longrightarrow$ diam $\sigma \leq \delta k$ for $k = 0, 1, 2, \ldots$.

To clarify the definition we give below the illustration of triangulation of G with C for $n = 1$.

2.4. EXAMPLE. For $n = 1$.

The characteristics of simplices of G with crogs is proved by Todd (1976) in follo Lemma.

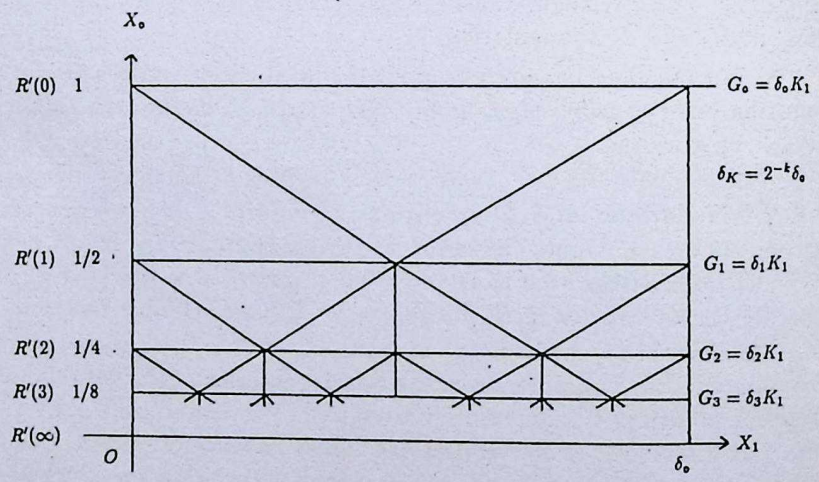


Fig 1

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

LEMMA. 2.5. *If G is triangulation with crogs and $\sigma \in G$ be any arbitrary simplex, then $\sigma \subseteq [2^{-k-1}, 2^{-k}] \times R^n$ for some $k \geq 0$.*

From the above example and the lemma we observe that a natural way to construct such triangulation is to choose $G_k = \delta k/n^{1/2}$ and then join up the G_k's suitably, where K_1 and J_1 are the triangulation of S^n and R^n, so we first define the triangulation K_1 due to Kuhn (1960) and J_1 of R^n is originally due to Tucker and also found in Whitney (1957).

2.6. The Triangulation of $\mathbf{K_1}$ of $\mathbf{R^n}$. Let $K_1^o = Z^n = \{y \in R^n | y^i \in Z$ for each $i \in N\}$. If $y^o \in K_1^o$ and π is a permutation of N, then n-simplex $< y^o, \ldots y^n >$ of such triangulation is denoted by $k_1(y^o, \pi)$, where $y^i = y^{i-1} + u^{\pi(i)}$ for each $i \in N$.

Finally, the collection of all such $k_1(y^o, \pi)$ whenever we set $k_1(y^o, \pi) =< y^o, \ldots y^n >$ and y^i's are ordered as above is called K_1-triangulation of R^n.

2.7. The triangulation $\mathbf{J_1}$ of $\mathbf{R^n}$. Let $J_1^o = Z^n$. Then the subset $J_1^{oc} = \{y \in J_1^o | y_i$ is odd for each $i \in N\}$ is called set of central vertices of J_1^o.

If $y^o \in J_1^{oc}, \pi$ is a permutation of N, and $s \in R^n$ is a sign vector (each s_i is ± 1), then the n-simplex $< y^o, \ldots, y^n >$ of this triangulation is denoted by $j_i(y^o, \pi, s)$, where $y^i - y^{i-1} + s_{\pi(i)} u^{\pi(i)}$ for each $i \in N$.

Then the collection of all such $j_1(y^o, \pi, s)$ is called J_1-triangulation. To obtain the triangulation of $K_2(m)$ and $J_2(m)$ first we define $\widetilde{K}_1$ and $\tilde{J}_1$.

2.8. The triangulation of $\widetilde{\mathbf{K}}_1$ and $\tilde{\mathbf{J}}_1$ of $\mathbf{S^n}$. For any compact convex set $C \subseteq R^n$ and $\delta \in R$, let $\delta C = \{\delta c | c \in C\}$.

Again, for any family G of subset of R^n and for any $0 \neq \delta \in R$, let $\delta G = \{\delta C | C \in G\}$, so that when G is a triangulation of $D \Longrightarrow \delta G$ is a triangulation of δD.

In particular, they give $\delta K_1, \delta J_1$ of R^n for any arbitrary mesh. They generate the triangulation of S^n by taking $C^n = \{X \in R^n : 1 \geq X_1 \geq \ldots \geq X_n \geq 0\}$.

Choose $0 < m \in Z$ and let $\widetilde{K}_1$ and $\tilde{J}_1$ be the collection of n-simplices of K_1 & J_1 meeting. $mC^n = \{X \in R^n | m \geq X_1 \geq \ldots \geq X_n \geq 0\}$, then $\widetilde{K}_1$ and $\tilde{J}_1$ triangulates mC^n.

The affine homeomorphism h on $m^{-1}\widetilde{K}_1$ and $m^{-1}\tilde{J}_1$ of S^n gives $K_2(m)$ and $J_2(m)$. Thus, we have

2.9. The triangulation $\mathbf{K_2(m)}$ of $\mathbf{S^n}$. Let Q denote $(n+1)Xn$ matrix

$$\begin{bmatrix} -1 & 0 & \ldots & 0 \\ +1 & -1 & \ldots & 0 \\ 0 & +1 & \ldots & \vdots \\ 0 & 0 & \vdots & -1 \\ \vdots & \vdots & \ldots & +1 \end{bmatrix}$$

and q^j its jth column for $j \in N$.

Let $K_2^o(m) = \{y \in S^n | my_i \in Z\}$ for each $i \in N_o$. If $y^o \in K_2^o(m)$ and π is a permutation of N, let $\sigma =< y^o, \ldots, y^n >$ denoted by $k_2(y^o, \pi)(m$ is implicit), where $y^i = y^{i-1} + m^{-1} q^{\pi(i)}$ for each $i \in N$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Finally, let $K_2(m)$ be the collection of all such $k_2(y^o, \pi)$.

2.10. The triangulation $\mathbf{J_2(m)}$ of $\mathbf{S^n}$. Let $J_2^o(m) = K_2^o(m)$ and $J_2^{oc} = \{y \in J_2^o(m) | my_i$ is even, $1 \leq i < n$ and my_n is odd$\}$. If $y^o \in J_2^{oc}(m), \pi$ is a permutation of N, and $s \in R^n$ a sign vector, then $\sigma =< y^o, \ldots, y^n >$, where $y^i = y^{i-1} + m^{-1} s_{\pi(i)} q^{\pi(i)}$ for each $i \in N$.

If $\sigma \subseteq S^n$, then we have $\sigma = j_2(y^o, \pi, s)$.

Finally, let $J_2(m)$ be the collection of all such $j_2(y^o, \pi, s)$.

The another characteristics of triangulation with crogs is given by the following lemma which has been proved by Todd (1976).

LEMMA. 2.11 *Let G be a triangulation with crogs so that $mesh_2' G_{k+1} | mesh_2' G_k < 1/2$ for some k. Then there exists $x \in R^n$ such that $[2^{-k-1}, 2^{-k}] \times \{x\}$ intersects more than $n+1$ simplices of G.*

Now taking $\epsilon_k = 2^{-k}$, so that $\delta_k = 2^{-k_n 1/2}$ and applying J_1, K_1 along with lemma (2.11) the triangulation with crogs called as J_3 and K_3 can be constructed.

Thus, these triangulations can be defined as follows:

2.12. The triangulation of $\mathbf{J_3}$. Let $y \in J_3^{oc}$ with $y_o \leq 1/2, \pi$ be a permutation of N_o with $\pi(j) = 0$ and $s \in R^n$ be a sign vector. Let $w = w(y)$. Then $j_3(y, \pi, s)$ is the simplex $\sigma =< y^{-1}, \ldots, y^n >$, where

$$y^{-1} = y; \tag{2.12.1}$$

$$y_i = y^{i-1} + y_o s_{\pi(i)} v^{\pi(i)}, 0 \leq i < j; \tag{2.12.2}$$

$$y^j = y^{j-1} - y_o \sum_{\ell=j}^{n} w_{\pi(\ell)} v^{\pi(\ell)}; \tag{2.12.3}$$

$$\text{and} \quad y^k = y^{k-1} + 2y_o w_{\pi(k)} v^{\pi(k)}, j < k \leq n \tag{2.12.4}$$

Thus, J_3 is the collection of all such $j_3(y, \pi, s)$.

In case σ is independent of $S\pi(k), j < k \leq n$, then such co-ordinate of s are indicated by dots.

2.13. The triangulation of $\mathbf{K_3}$. Let $y \in K_3^o$ with $t = t(y)$ and let π be a permutation of N_o with $\pi(j) = 0$. We call (y, π) admissible if $t_{\pi(k)} = 0$ for all $j < k \leq n$. In this case $k_3(y, \pi)$ is the simplex $< y^{-1}, \ldots, y^n >$, where

$$y^{-1} = y; \tag{2.13.1}$$

$$y^i = y^{i-1} + y_o v^{\pi(i)}, \quad 0 \leq i < j \tag{2.13.2}$$

$$y^i = y^{i-1} + y_o \left(-\sum_{i=0}^{j-1} t_{\pi(i)} v^{\pi(i)} + v^o - \sum_{k=j+1}^{n} v^{\pi(k)} \right) \tag{2.13.3}$$

$$y^k = y^{k-1} + 2y_o v^{\pi(k)}, \quad j < k \leq n. \tag{2.13.4}$$

Finally, $K_3 = \left\{ k_3(y, \pi) \middle| (y, \pi) \text{ admissible} \right\}$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

2.14. REMARK. It is to be observed that every collection of simplices $< y^{-1}, \ldots, y^n >$ of K_3 may not give a triangulation. This can be seen by the following example for $n = 1$.

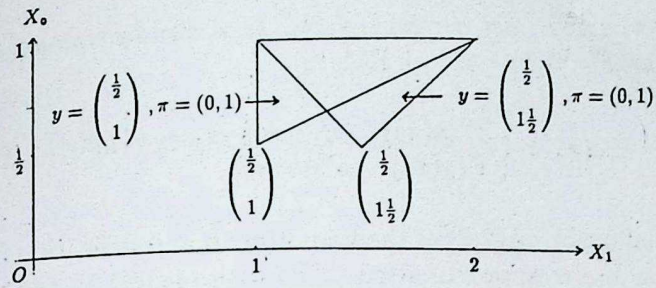


Fig 2

The comprehensive understanding of J_3 and K_3 can be found in the lecture notes due to Todd (1976) and in his paper (Todd 1976).

Since the algorithm is based on the homotopy and vector labelling, so we will explain the vector labelling with respect to upper semi-continuous and piecewise linear approximation to u.s.c.

2.15. Vector Labelling and Complete simplices for $\mathbf{J_1}$. Let A be affine subspace in $R^m, m > n$ such that $\dim A = n$. Let $F : A \to A^*$ be the u.s.c. and G be the triangulation of A, and let f be piecewise linear approximation to F with respect to G and S be the linear subspace of A, then $\ell : A \to S$ defined by

$$\ell(x) = f(x) - x \tag{2.15.1}$$

is called the vector labelling for f (or a vector labelling for F with respect to G).

If $\sigma =< y^o, \ldots, y^k > \in G^k$ then the $(n+1) \times (k+1)$ matrix given by

$$L_\sigma = \begin{bmatrix} 1 & \ldots & 1 \\ \ell(y^o) & \ldots & \ell(y^k) \end{bmatrix} \tag{2.15.2}$$

is called the label matrix of σ.

Let t^o be the unit vector $(1, 0, 0, \ldots, 0)^T$ in R^{m+1} then the fixed point of f by means of ℓ are given by the following lemma proved in Todd (1976).

LEMMA. 2.16. *Let A, F, G, f amd ℓ be defined as above and let $\sigma =< y^o, \ldots, y^n > \in G$, then there is a fixed point of f in $\bar{\sigma}$ if and only if there is a solution to*

$$L_\sigma W = t^o \in R_+^{n+1} \tag{2.16.1}$$

if W solved (2.16.1) and we have $X^* = \Sigma_{i \in N_o} w_i y^i$ then X^* is a fixed point of f.

DEFINITION. 2.17. *A simplex σ of G is said to be complete if there is a solution to* (2.16.1).

In fact the vector labelling defined in (2.15) is a homotopy from A to a piecewise linear approximation to F_1. Where F_1 is u.s.c. mapping from R^n to R^{n*}. The zeros of this

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

function ℓ are the fixed points of a homotopy from a given linear function to the function whose approximate fixed points are to be determined, so using the triangulation G of (2.3) we define a homotopy h from a linear function f^o to f.

DEFINITION. 2.18. *Let $a : R_+ \to R_+$ be a nondecreasing function and let $f^o(x) = A(c-x)+x$. Define $F' :]0,1] \times R^n \to R^{n*}$ by*

$$F'(x) = \begin{cases} f^o(p(x)) & \text{if } -\log_2 x_o \leq a(||p(x)-c||_2) \\ F(p(x)) & \text{otherwise} \end{cases} \tag{2.18.1}$$

From the definition of (2.2) of the homotopy h and definition of (2.18) one can conclude that h is a piecewise linear approximation to F' with respect to the triangulation G.

By setting $h : [0,1] \times R^n \to R^n$ as

$$h(0, X^T)^T = f(x). \tag{2.18.2}$$

Todd (1976) has shown in the following lemma that h is homotopy.

LEMMA. 2.19. *h is the homotopy from f to f^o defined by* (2.18.2).

Corresponding to this homotopy the fixed point is defined as follows:

DEFINITION. 2.20. *The point $X \in [0,1] \times R^n$ is a fixed point of h if $h(x) = p(x)$. Moreover $x \in R^n(0)$ is a fixed point of h iff $p(x) = c$ or $x = (1, c^T)^T$ when A is non-singular.*

2.21. REMARK. A point $x \in R^n(\infty)$ is a point of h iff $p(x)$ is a fixed point of F where F is u.s.c. such that $F = \{f\}$.

In case $F \neq \{f\}$ then x is not the fixed point of F.

The characterization of the set of the fixed point of homotopy h is proved by Todd (1976) in the following lemma:

LEMMA. 2.22. *Let $\{x^k\}$ be a sequence of fixed point of h tending to $x \in R^n(\infty)$. Then $p(x)$ is a fixed point of u.s.c. mapping F.*

Moreover we know that the fixed points is ultimately obtained where there is a completely labelling simplices, so to have the complete simplices or very complete simplices for crogs the vector labelling ℓ in definition (2.15) is modified as $[0,1] \times R^n \to R^n$ given by $\ell(x) = h(x) - p(x)$, where $\sigma \in G^n \cup G^{n+1}$ and the label matrix of σ is given by

$$L_\sigma = \begin{bmatrix} 1 & \dots & 1 \\ \ell(y^{-1}) & \dots & \ell(y^k) \end{bmatrix}, \tag{2.22.1}$$

where $\sigma = < y^{-1}, \dots, y^k >$.

According the condition of the fixed point of homotopy h given by the following lemma due to Todd (1976).

LEMMA. 2.23. *Let $\sigma \in G^n \cup G^{n+1}$. Then there is a fixed point of h in $\bar{\sigma}$ iff σ is complete. If $\sigma \in G_o$ is complete.*

Then $(1, c^T)^T \in \bar{\sigma}$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

3. On applying all the above stated conditions for the homotopy h, the step of algorithm can be constructed as follows to obtain the fixed point of homotopy h in $R^n(k)$ for every k.

3.1. To Algorithm. Let Γ be the infinite path of the mapping whose approximate fixed point is to be determined. Then we have the following steps:

STEP 0. Let σ_1 be the unique $(n+1)$-simplex of G with τ_0 as a facet. Let y^+ be the vertex of σ_1 outside $R^n(0)$ from the lexicographic linear system so that τ_0 is very complete. Setting $m \leftarrow 1$.

STEP 1. Calculate $\ell(y^+)$, introduce the column $\begin{pmatrix} 1 \\ \ell(y^+) \end{pmatrix}$ into the current basis of the lexicographic linear system. Assuming that the column $\begin{pmatrix} 1 \\ \ell(y^-) \end{pmatrix}$ drops from the basis with y^- a vertex of σ_m.

STEP 2. Let τ_m be the very complete facet of σ_m opposite y^-. Calculate the fixed point of the homotopy in $\bar{\tau}_m$ and terminate if it is sufficiently accurate. Otherwise let σ_{m+1} be the simplex of G sharing the facet of τ_m with σ_m. Let y^+ be the new vertex, on setting $m \leftarrow m+1$ return to step 1.

Generally, the fixed point of h in $\bar{\tau}_m$ is calculated only when $\tau_m \subseteq R^n(k)$ for some k.

The theoretical result based on the algorithm (3.1) can be seen in our joint paper (Shukla and Bawa 1990). So the computation procedure required in this paper particular for K_3 type simplices is given below:

3.2. To obtain the simplices of triangulation due to K_3 (where x_0 cannot be expressed on some power of 1/2).

We will first pick up element

$$\Gamma \in]y_0 - x_0, y_0]. \tag{3.2.1}$$

Let any component y^ℓ of simplex $\sigma =< y^{-1}, \ldots, y^n >$ be denoted by $z = z(\Gamma)$, where

$$z_i = x_i - y_1 \tag{3.2.2}$$

and

$$w_i = z_i + z_0 \mu_i \tag{3.2.3}$$

where

$$\mu_i = 1 - y_i/y_0 \bmod 2. \tag{3.2.4}$$

We consider the following two cases:

CASE I. Suppose $\Gamma > x_0 - y_0$ then z can be obtained as follows

$$z_i = \begin{cases} x_i \text{ if } x_i/y_0 \in E \text{ where } E \text{ stand for even number} \\ x_i + x_0 - y_0 \text{ if } x_i + x_0 - y_0/y_0 \in 0 \text{ where } 0 \text{ stand for odd number.} \end{cases} \tag{3.2.5}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In case (3.2.5) fails then y_i and w_i are obtained from x as follows:

$$z_i = \begin{cases} y_i + y_o & \text{if } x_i \geq \Gamma \text{ and} \\ y_i & \text{if } w_i < \Gamma \text{ (this case also include } i = 0) \end{cases} \tag{3.2.6}$$

CASE II. Suppose $\Gamma \leq x_o - y_o$ then z can be obtained by taking

$$z_i = \begin{cases} x_i & \text{if } x_i/y_o \in E \\ x_i + x_o - 2y_o & \text{if } x_i + x_o - y_o/y_o \in 0 \end{cases} \tag{3.2.7}$$

If (3.2.7) fails then y_i and w_i are determined from x as follows

$$z_i = \begin{cases} y_i + y_o & \text{if } w_i \geq \Gamma \text{ (this case also include } i = 0) \\ y_i - y_o & \text{if } w_i < \Gamma \end{cases} \tag{3.2.8}$$

4. Main Result. First we will consider J_3 triangulation of $]0,1] \times R^n$ for $n = 3$.

4.1. Suppose $x = (3/8, 1/2, -5/4, 5/8)^T$ where $x_o = 3/8 \Rightarrow 1/4 < x_o < 1/2$. (4.1.1)

From the position x_o we can have two choices from element $y \in \sigma$ a simplex of J_3 given by

$$y = (1/4, 1/4 - 5/4, 3/4)^T \tag{4.1.2}$$

$$\text{or,} \quad y = (1/4, 3/4, -5/4, 3/4)^T \tag{4.1.3}$$

we will cosider (4.1.3) where $y_o = 1/4$ from $w_i(y)$ equal to -1 or $+1$ according to whether y_i/y_o is 1 or 3 modulo 4, we have

$$w(y) = (-1, +1, +1, +1) \tag{4.1.4}$$

on solving $x_i - y_i$ for $i = 0, 1, 2, 3$ we have $x_o - y_o = 1/8, x_1 - y_1 = -1/4, x_2 - y_2 = 0$ and

$$x_3 - y_3 = -1/8 \tag{4.1.5}$$

using (4.1.4) and (4.1.5) we have

$$y_o \geq -1(x_1 - y_1) \geq -1(x_3 - y_3) \geq (x_o - y_o) \geq +1(x_2 - y_2) \geq (y_o - x_o)$$

$$\Rightarrow \quad 1/4 \geq 1/8 \geq 1/8 \geq 0 \geq -1/8 \tag{4.1.6}$$

$$\Rightarrow \quad \pi_4 = (1, 3, 0, 2) \quad \text{and} \quad s = (+1, \ldots, -1) \tag{4.1.7}$$

$$\text{or,} \quad y_o \geq -1(x_1 - y_1) \geq x_o - y_o \geq +1(x_2 - y_2) \geq +1(x_3 - y_3) \geq (y_o - x_o) \tag{4.1.8}$$

$$\Rightarrow \quad 1/4 \geq 1/4 \geq 1/8 \geq 0 \geq -1/8 \geq -1/8 \tag{4.1.9}$$

$$\Rightarrow \quad \pi_4 = (1, 0, 2, 3) \quad \text{and} \quad s = (-1, +1, \ldots). \tag{4.1.10}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Suppose $\sigma = j_3(y, \pi, s) =< y^{-1}, y^o, y^1, y^2, y^3 >$ then using definition (2.12) we have

$$y^{-1} = y = (1/4, 3/4, -5/4, 3/4)^T \tag{4.1.11}$$

$$y^o = (1/4, 3/4, -5/4, 3/4)^T + 1/4(-1)(0,1,0,0) = (1/4, 1/2, -5/4, 3/4)^T \tag{4.1.12}$$

$$y^1 = (1/4, 1/2, -5/4, 3/4)^T + 1/4(-1)(0,0,0,1) = (1/4, 1/2, -5/4, 1/2)^T \tag{4.1.13}$$

$$y^2 = (1/4, 1/4, -5/4, 1/2)^T - 1/4(-1)(1,0,0,0,0) + (0,0,1,0)$$
$$= (1/2, 1/2, -3/2, 1/2)^T \tag{4.1.14}$$

$$y^3 = (1/2, 1/2, -3/2, 1/2)^T + 2.1/4(0,0,1,0) = (1/2, 1/2, -1, +1/2)^T \tag{4.1.15}$$

Denoting $a_{-1} = y_o$ and $a_i = w_i(y)(x_i - y_i)$ for $0 \le i \le j$ and $a_j = x_o - y_o$ again $a_k = w_k(y)(x_k - y_k)$ for $j < k \le n$ and $a_{n+1} = y_o - x_o a_{-1} = 1/4, a_o = 1/4, a_1 = 1/8, a_2 = 1/8, a_3 = 0, a_4 = -1/8$

Defining β by $\beta_i = a_i - a_{i+1}/y_o$ for $-1 \le i < j$ and $\beta_k = (a_k - a_{k+1})/2y_o$ we have $\beta_{-1} = 0, \beta_o = 1/2, \beta_1 = 0, \beta_2 = 1/4, \beta_3 = 1/4$ from these we have

$$x = \sum_{i=0}^{3} \beta_i y^i = 1/2y^o + 1/4y^1 + 1/4y^2.$$

4.2. Again if we take $x_o = 3/10$ where $1/4 < x_o < 1/2$ in (4.1) then, again, one can easily show that the simplices of the triangulation J_3 can be considered same as (4.1.2) and (4.1.3) on considering (4.1.2.), we have

$$w(y) = (-1, -1, +1, +1). \tag{4.2.1}$$

On solving $x_i - y_i$ for $i = 0, 1, 2, 3$ we have

$$x_o - y_o = 1/20, x_1 - y_1 = 1/4, x_2 - y_2 = 0, x_3 - y_3 = 1/8 \tag{4.2.2}$$

using (4.2.1) and (4.2.2) we have

$$y_o \ge +1(x_1 - y_1) \ge -1(x_3 - y_3) \ge x_o - y_o \ge +1(x_2 - y_2) \ge y_o - x_o$$
$$1/4 \ge 1/4 \ge 1/8 \ge 1/20 \ge 0 \ge -1/20 \tag{4.2.3}$$
$$\Rightarrow \quad \pi_4 = (1, 3, 0, 2) \quad \text{and} \quad s = (+1, -1, +1) \tag{4.2.4}$$

than using definition (2.12) we have

$$y^{-1} = (1/4, 1/4, -5/4, 3/4)^T \tag{4.2.5}$$
$$y^o = (1/4, 1/2, -5/4, 3/4)^T \tag{4.2.6}$$
$$y^1 = (1/4, 1/2, -5/4, 1/2)^T \tag{4.2.7}$$
$$y^2 = (1/2, 1/2, -3/2, 1/2)^T \tag{4.2.8}$$
$$y^3 = (1/2, 1/2, -1, 1/2)^T \tag{4.2.9}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Here again,

$$a_{-1} = 1/4, a_o = 1/4, a_1 = 1/8, a_2 = 1/20, a_3 = 0, a_4 = -1/20 \tag{4.2.10}$$

$$\Rightarrow \quad \beta_{-1} = 0, \beta_o = 1/2, \beta_1 = 3/10, \beta_2 = 1/10, \beta_3 = 1/10 \tag{4.2.11}$$

$$\Rightarrow \quad x = \sum_{i=0}^{3} \beta_i y^i = 1/2y^o + 3/10y^1 + 1/10y^2 + 1/10y^3.$$

We observe from the (4.1) and (4.2) that so long as the pivot element $x_o \in]2^{-k}, 2^{-k+1}[$ for some arbitrary but fixed k, then we will have some set of simplices.

In case if we change k then we will have different set of simplices of the triangulation due to j_3. To prove it let us consider the following.

4.3. $x = (3/16, 1/4, -5/8, 5/16)^T$ where

$$x_o = 3/16 \quad \Rightarrow \quad 1/8 < x_o < 1/4. \tag{4.3.1}$$

In this case, we have,

$$y = (1/8, 1/8, -5/8, 3/8)^T \tag{4.3.2}$$

We have,

$$w(y) = (-1, -1, +1, +1) \tag{4.3.3}$$

On solving $x_i - y_i$ for $i = 0, 1, 2, 3$.

We have,

$$x_o - y_o = 1/16, x_1 - y_1 = 1/8, x_2 - y_2 = 0, x_3 - y_3 = -1/16 \tag{4.3.4}$$

Using (4.3.3) and (4.3.4), we have,

$$y_o \geq +1(x_1 - y_1) \geq -1(x_3 - y_3) \geq x_o - y_o \geq 1(x_2 - y_2) \geq y_o - x_o \tag{4.3.5}$$

$$1/8 \geq 1/8 \geq 1/16 \geq 1/16 \geq 0 \geq -1/16 \tag{4.3.6}$$

$$\pi_4(1, 3, 0, 2) \quad \text{and} \quad s = (+1, ., -1) \tag{4.3.7}$$

then using definition (2.12), we have,

$$y^{-1} = (1/8, 1/8, -5/8, 3/8)^T \tag{4.3.8}$$

$$y^o = (1/8, 1/4, -5/8, 3/8)^T \tag{4.3.9}$$

$$y^1 = (1/8, 1/4, -5/8, 1/4)^T \tag{4.3.10}$$

$$y^2 = (1/4, 1/4, -3/4, 1/4)^T \tag{4.3.11}$$

$$y^3 = (1/4, 1/4, -1/2, 1/4)^T \tag{4.3.12}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

we have,
$$a_{-1} = 1/8, a_o = 1/8, a_1 = 1/16, a_2 = 1/16, a_3 = 0, a_4 = -1/16. \tag{4.3.13}$$

We will get
$$\beta_{-1} = 0, \beta_o = 1/2, \beta_1 = 0, \beta_2 = 1/4, \beta_3 = 1/4 \tag{4.3.14}$$

From these, we have,

$$x = \sum_{i=0}^{3} \beta_i y^i = 1/2y^o + 1/4y^2 + 1/4y^3$$

4.4. If we now consider the triangulation of K_3 for the same set of x and x_o as a pivot element as given in (4.1) and (4.3) we will show that it will give three simplices.

4.4.1. Again taking $x_o = 3/8, y_o = 1/4$ and applying (3.2) we have

$$\Gamma \in (y_o - x_o, y_o] \Rightarrow \Gamma \in (-1/8, 1/4)$$

Suppose $\Gamma = 3/16 \Rightarrow 1/8 < \Gamma \leq 1/4$.

Since $3/16 > 1/8 \Rightarrow \Gamma > x_o - y_o$.

Using case I of (3.2) we observe on applying (3.2.5) for $i = 0$, we have $x_o/y_o = 3/2 \notin E$ and $x_o + x_o - y_o/y_o = 2 \notin 0 \Rightarrow$ (3.2.5) for z_o is not satisfied.

Applying (3.2.3), (3.2.4), (3.2.6), we have, $w_o = 1/8 < 3/16\mu_o = 1 - 1/4/1/4\text{mod } 2 = 1 - 1\text{mod } 2 = 0\text{mod } 2 \Rightarrow \quad z_o = y_o = 1/4$.

Similarly, other component z_1, z_2, z_3 are obtained as follows $z_1 = 1/2, z_2 = -5/4, z_3 = 3/4$.

These give the simplex $y_o = (1/4, 1/2, -5/4, 3/4)^T$.

Again taking $\Gamma = 1/16 \Rightarrow 0 < \Gamma \leq 1/8$.

Since, $1/16 < 1/8 \Rightarrow \Gamma < x_o - y_o$.

Using case II of (3.2) we observe on applying (3.2.7) let $i = 0$, we have, $x_o/y_o = 3/2 \notin E$, and $x_o + x_o - y_o/y_o = 2 \notin 0$ (3.2.7) for z_o is not satisfied, so applying (3.2.3), (3.2.4) and (3.2.8), we have, $w_o = 1/8 > 1/16 \Rightarrow \mu_o = 0$ then $z_o = y_o + y_o = 1/4 + 1/4 = 1/2$

Similarly, other component z_1, z_2, z_3 are obtained as follows: $z_1 = 1/2, z_2 = -3/2, z_3 = 1/2$.

They give the simplex $y^2 = (1/2, 1/2, -3/2, 1/2)^T$.

Suppose $\Gamma = -1/16 \Rightarrow -1/8 < \Gamma \leq 0$.

Since $-1/16 < 1/8 \Rightarrow \Gamma < x_o - y_o$.

Using case II of (3.2) we observe on applying (3.2.7) for $i = 0; x_o/y_o = 3/2 \notin E$ and $x_o + x_o - y_o/y_o = 2 \notin 0$ (3.2.7) for z_o is not satisfied on applying (3.2.3), (3.2.4) and (3.2.8), we have, $w_o = 1/8 > -1/16 \Rightarrow \mu = 0 \Rightarrow z_o = 1/2$.

Similarly, other component z_1, z_2, z_3 are obtained as follows $z_1 = 1/2, z_2 = -1, z_3 = 1/2$ we have simplex y^3 give by $y^3 = (1/2, 1/2, -1, 1/2)^T$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Which shows that one gets a set of three simplices due to crogs triangulation K_3 of $]0,1] \times R^n$, while it gives a set of 4 triangulate due to J_3.

By using the algorithm of (3.1), the fixed point of any piecewise linear function can be obtained from simplices $n > 3$ by constructing a computer programming.

References

Eaves, B. C. (1972) : "Homotopies of computation of fixed points", *Mathematical Programming, (3) 1*,1.

Eaves, B. C. and **Saigal, R.** (1972) : "Homotopies for computation of fixed points on unbounded Regions", *M. Pro. (3)* 2, 225.

Freudenthal, H. (1942) : "Simplizialzerlegungen Von-Beschrankter Flachheit *Annal of Mathematics", (3) 43*, 580.

Kuhn, H. W. (1960) : "Some Combinatorial Lemmas in topology", *IBM J. Research and Develop, (4) 5*, 518.

Lemke, C. E. (1970) : *"Recent Results on Complementarity Problems" Nonliner Programming Ed.* J. B. Rosen, O.L. Mangasarian, and K. Ritter, Academic Press, New York 349.

Shukla, D. P. and **Bawa N. P. S.** (1990) : "$\widetilde{K}$'3–A new triangulation of R^n", *Proc. Indian Acad. Sci.* (*Math. Sci.*), *(100) 1*, 79.

Todd, M. J. (1976) : "On Triangulation for Computing Fixed Points", *Math. Programming* (10), 322.

Todd, M. J. (1980) : "Traversing Large Piece of Linearity in Algorithms that solve equations by following Piecewise-Linear Paths", *Mathematics of Operation Research* (5) 2.

Todd, M. J. (1976) : *"The computation of fixed points and applications" Lecture Notes in Economics and Mathematical Systems*, 124, Springer-Verlag, New York.

Whitney, H. (1957) : *"Geometric Integration Theory"*, Princeton University Press, Princeton.

DEPARTMENT OF MATHEMATICS
GOVT. MODEL SCIENCE COLLEGE
REWA 486001 M.P.
INDIA

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc. **86**, 373-378 (1994)

ON AXISYMMETRIC WAVES IN WATER WITH AN INERTIAL SURFACE

N. K. GHOSH

(*Received 25 February 1990*)

Abstract : In this paper we derive the velocity potential due to a submerged circular ring source of time-dependent strength outside an immersed vertical coaxial circular cylinder in a liquid of finite constant depth with an inertial surface composed of uniformly distributed noninteracting floating particles under the influence of surface tension at the inertial surface. Laplace transform in time and modified Weber transform in radial co-ordinate has been used to obtain the velocity potential. For time-harmonic source strength, the steady-state development shows the existence of outgoing progressive waves under any inertial surface and in the absence of surface tension such waves cannot propagate if the inertial surface is too heavy.

1. Introduction : Assuming linearised theory of water waves and irrotational motion, various problems on the generation of infinitesimal gravity waves in a motionless liquid of both finite and infinite depths with an inertial surface composed of a thin uniform distribution of disconnected floating matter (e.g. broken ice, unstretched mat) were considered by Rhodes-Robinson (1984), Mandal and Kundu (1986, 1987, 1988). After using Laplace transform in time, these problems were solved in a manner similar to those used for the corresponding time harmonic problems with a free surface.

The study of generation of surface waves in the presence of a vertical body of revolution partially immersed or submerged in a liquid requires the consideration of potentials due to submerged circular horizontal ring of wave sources when the body and the liquid region have a common axis of symmetry as the problems can be formulated in terms of suitable distribution of rings of wave sources around the body (Hulme (1981), Fenton (1978)). An offshore structure in the high sea for the purpose of oil prospecting may be modelled as a vertical cylinder of circular cross section. Hence consideration of velocity potential due to submerged circular rings of wave sources outside an immersed coaxial circular cylinder is of some importance. For a liquid with a free surface in the presence of surface tension Rhodes-Robinson (1979) obtained the ring source potential outside such a coaxial circular cylinder while considering a class of time-harmonic surface wave problems involving immersed vertical boundaries. Later Mandal and Kundu (1988) extended this problem for a liquid with an inertial surface in the absence of surface tension.

In the present paper, we consider a ring source submerged in a liquid of uniform finite depth with an inertial surface in the presence of surface tension outside an immersed coaxial circular cylinder. After taking Laplace transform in time, recently Mandal and Kundu (1989) solved the cylindrical wave-maker problem by using a modified form of Weber transform formula in the radial coordinate. The same transformation formula has been employed here to obtain the transformed velocity potential after taking Laplace transform in time. Laplace inversion then produces the velocity potential. As a special case, we consider a ring source of time harmomic strength $m(t) = \sin \sigma t$ and obtain the steady-state development of the potential function for large time following a prodcedure used by Rhodes-Robinson (1984). It is found that disturbances propagate at large distance from the ring source for all frequency so long as the surface tension is not zero. In the absence of the inertial surface and tension known results are recovered by making the depth of the liquid very large.

2. Statement and formulation of the problem : We consider the motion of an incompressible inviscid liquid of finite uniform depth h and of volume dentity ρ under the action of gravity and surface tension, the surface of which is covered by an inertial surface composed of a thin uniform distribution of disconnected heavy floating matter of area density $\rho\varepsilon \geq 0$. $\varepsilon = 0$ corresponds to a liquid with a free surface. A horizontal ring with radius b of

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

uniformly distributed sources, each of the same strength $m(t)$, is present in the liquid at a depth u (> 0) below the mean inertial surface outside an immersed coaxial circular cylinder of radius a (< b). We choose a cylindrical coordinate system (r, θ, y) in which the y axis is taken vertically downwards as the axis of the ring. The plane $y = 0$ corresponds to the undisturbed position of the inertial surface and the liquid occupies the region $0 \le y \le h$ with horizontal bottom $y = h$. The vertical circular cylinder is extended throughout the depth of the liquid and the ring is situated at $r = b$ (> a), $y = u$ (> 0). We assume the ring sources to commence operating at time $t = 0$ and the motion of the liquid is axisymmetric as the strength of the ring is assumed to be uniform in θ. The motion is irrotational since it is generated from rest and may be described by a velocity potential $\varphi(r, y; t)$ for $t > 0$ which satisfies the Laplace's equation

$$\varphi_{rr} + \frac{1}{r}\varphi_r + \varphi_{yy} = 0, \qquad \text{..... (2.1)}$$

in the liquid region except at points on the ring. Within the frame-work of linearised theory φ satisfies the condition at the inertial surface (Mandal and Kundus 1987)

$$\frac{\partial^2}{\partial t^2}\left(\varphi - \varepsilon\varphi_y\right) - g\varphi_y - \frac{T}{\rho}\varphi_{yyy} = 0, \quad \text{on } y = 0 \qquad \text{..... (2.2)}$$

where g is the gravity and T is the co-efficient of surface tension, the bottom condition

$$\varphi_y = 0, \quad \text{on } y = h \qquad \text{..... (2.3)}$$

the condition on the cylinder and the edge condition

$$\varphi_r = 0, \quad \text{on } r = a, \ \varphi_{ry}\left(a^+, 0; t\right) = 0, \qquad \text{..... (2.4)}$$

and the initial conditions (Rhodes-Robinson 1984)

$$\frac{\partial}{\partial t}\left(\varphi - \varepsilon\varphi_y\right) = \varphi - \varepsilon\varphi_y = 0 \quad \text{on } y = 0, t = 0. \qquad \text{..... (2.5)}$$

Let $\overline{\varphi}(r, y; p)$ denote the Laplace transform of $\varphi(r, y; t)$ defined as

$$\overline{\varphi}(r, y; p) = \int_0^\infty \varphi(r, y; t) e^{-pt} dt (p > 0).$$

Then the initial value problem described by (2.1) to (2.5) reduces to the following boundary value problem

$$\overline{\varphi}_{rr} + \frac{1}{r}\overline{\varphi}_r + \overline{\varphi}_{yy} = \frac{\delta(r-b)\delta(y-u)}{2\pi r}\overline{m}(p),$$

$$a < r < \alpha, \ 0 < y < h \qquad \text{..... (2.6)}$$

where $\overline{m}(p)$ is the Laplace transform of $m(t)$,

$$p^2\overline{\varphi} - \left(g + \varepsilon p^2\right)\overline{\varphi}_y - \frac{T}{\rho}\overline{\varphi}_{yyy} = 0, \quad \text{on } y = 0, \ r > a, \qquad \text{..... (2.7)}$$

$$\overline{\varphi}_y = 0 \text{ on } y = h \qquad \text{..... (2.8)}$$

and

$$\overline{\varphi}_r = 0 \text{ on } r = a, \ \overline{\varphi}\left(a^+, 0; p\right) = 0 \qquad \text{..... (2.9)}$$

3. Solution of the BVP : We use the following modified form of the Weber transform of a function $g(r)$ defined in (a, ∞) by

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$G(k) = \int_a^\infty rA(r,k)\,g(r)\,dr \qquad \text{..... (3.1)}$$

where $A(r,k) = J_1(ak)Y_0(rk) - J_0(rk)Y_1(ak)$ ..... (3.2)

and J_n, Y_n ($n = 0, 1$) are the Bessel functions of the first and second kinds respectively. The inverse transform formula is

$$g(r) = \int_0^\infty \frac{kA(r,k)G(k)}{J_1^2(ak) + Y_1^2(ak)}\,dk. \qquad \text{..... (3.3)}$$

This modified form of the Weber transform has been used by Mandal and Kundu (1989) in connection with a cylindrical wave-maker problem. We note that

$$\int_a^\infty \left(g_{rr} + \frac{1}{r^{g}{}_{r}}\right) r\,A(r,k)\,dr = -k^2 G(k). \qquad \text{..... (3.4)}$$

Let $f_1(k,y)$ denote the Weber transform of $\overline{\varphi}(r,y)$. Then writing $f_1(k,y) = \dfrac{\overline{m}(p)A(b,k)}{2\pi} f(k,y)$ and using (3.4) the BVP described by (2.6) to (2.9) reduces to the simpler BVP given by

$$\frac{d^2 f}{dy^2} - k^2 f = \delta(y-u),$$

$$p^2 f - \left(g + \varepsilon p^2\right) f_y - \frac{T}{\rho} f_{yyy} = 0, \quad \text{on } y = 0, \qquad \text{..... (3.5)}$$

$$f_y = 0 \text{ on } y = h.$$

The solution of the BVP (3.5) is

$$f(k,y) = \frac{\cos hk(h-u)}{k}\left[\frac{\sin hky + \varepsilon k \cos hky}{D(k)} + \frac{\mu^2 \cos hk(h-y)}{\sin hkh D(k)\left(p^2+\mu^2\right)}\right], \quad 0 < y < u \qquad \text{..... (3.6)}$$

where $D(k) = \cos hkh + \varepsilon k \sin hkh$, $M = T/\rho g$.

and $$\mu^2 = \frac{gk\left(1 + Mk^2\right)\sin hkh}{D(k)}$$

for $0 < u < y$, we interchange y and u in the expression (3.6). Inverse Weber transform formula (3.3) gives

$$\varphi(r,y;p) = \frac{1}{2\pi}\int_0^\infty \frac{\overline{m}(p)A(r,k)A(b,k)}{J_1^2(ak) + Y_1^2(ak)}\left[\frac{\sin hky + \varepsilon k \cos hky}{D(k)} + \frac{\mu^2 \cos hk(h-y)}{\sin hkh D(k)\left(p^2+\mu^2\right)}\right] \cos hk\,(h-u)\,dk,$$

$0 < y < u$. ..... (3.7)

Taking Laplace inversion we obtain

$$\varphi(r,y;t) = \frac{1}{2\pi}\int_0^\infty \frac{A(r,k)A(b,k)\cos hk(h-u)}{J_1^2(ak) + Y_1^2(ak)}\left[\frac{m(t)(\sin hky + \varepsilon k \cos hky)}{D(k)} + \frac{\mu \cos hk(h-y)}{D(k)\sin hkh}\right.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\int_0^t m(\tau)\sin\mu\,(t-\tau)d\tau\Bigg]dk, 0<y<u. \qquad \text{..... (3.8)}$$

(3.8) is the general result for any arbitrary time-dependent strength $m(t)$ of the sources. By taking the particular forms of the source strength $m(t)$ as impulsive, constant or time-harmonic etc., the corresponding potential functions can be deduced from this general result.

4. Time-harmonic ring source and steady-state development : In this section we consider the special case where the strength of the sources varies harmonically with time. By taking $m(t) = \sin \sigma t$, we find from (3.8) that

$$\varphi(r,y;t)=\frac{1}{2\pi}\int_0^\infty \frac{A(r,k)A(b,k)\cos hk(h-u)}{J_1^2(ak)+Y_1^2(ak)}\Bigg[\frac{\sin\sigma t(\sin hky+\varepsilon k\cos hky)}{D(k)}$$

$$+\frac{\mu\cos hk(h-y)}{D(k)\sin hkh}\frac{\mu\sin\sigma t-\sigma\sin\mu t}{\mu^2-\sigma^2}\Bigg]dk,\ 0<y<u. \qquad \text{..... (4.1)}$$

To obtain the steady-state development in (4.1), we must isolate and eliminate the transients terms. The denominator of the second integral in (4.1) vanishes in the range of integration $k > 0$ if $\mu^2 - \sigma^2$ or equivalently $Q(k) = k(1 - K\varepsilon + Mk^2)\sin hkh - K\cos hkh$ (where $K = \sigma^2/g$) has a real zero for $k > 0$. For $M > 0$ and $\varepsilon K \geq 0$, $Q(k)$ has always two real zeros (Rhodes-Robinson 1971), one positive and another negative, k_0 and $-k_0$ say and an infinite number of imaginary zeros $\pm ik_n$ say, where k_n's ($n = 1,2, \ldots$) are real and positive and satisfy the equation

$$k_n\left(1-K\varepsilon-Mk_n^2\right)\sin k_nh+K\cos k_nh=0.$$

We now introduce a Cauchy principal value at $k = k_0$ and following a procedure used by Rhodes-Robinson (1984) we split the integral involving $\sin \mu t$ in (4.1) into two parts given by

$$-\frac{2\sigma}{\pi}\int_0^\infty\Bigg[\frac{A(r,k')A(b,k')\cos hk'(h-u)\cos k'(h-y)}{J_1^2(ak')+Y_1^2(ak')}$$

$$\cdot\frac{k'\left(1+Mk'^2\right)}{\left\{\left(1+3Mk'^2\right)\sin h2k'h+2k'h\left(1+Mk'^2\right)+4M\varepsilon k'^3\sin h^2k'h\right\}\left(\mu(k')+\sigma\right)}\Bigg]_{k'=k_0}^{k}\frac{\sin\mu t}{\mu-\sigma}d\mu$$

$$-\frac{1}{\pi}\frac{A(r,k_0)A(b,k_0)\cos hk_0(h-u)\cosh k_0(h-y)}{J_1^2(ak_0)+Y_1^2(ak_0)}$$

$$\cdot\frac{k_0\left(1+Mk_0^2\right)}{\left(1+3Mk_0^2\right)\sin h2k_0h+2k_0h\left(1+Mk_0^2\right)+4M\varepsilon k_0^3\sin h^2k_0h}\int_0^\infty\frac{\sin\mu t}{\mu-\sigma}\,d\mu \qquad \text{..... (4.2)}$$

It is evident that the first term of (4.2) is transient, by Reimann-Lebesgue lemma and the integral in the second term of (4.2) tends to $\pi \cos \sigma t$ as $t \to \infty$. Thus after simplification we finally obtain as $t \to \infty$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\varphi(r,y;t) \sim \frac{\sin\sigma t}{2\pi}\int_0^\infty \frac{A(r,k)A(b,k)\cos hk(h-u)}{J_1^2(ak)+Y_1^2(ak)}\cdot\frac{k\left(1-K\varepsilon+Mk^2\right)\cos hky-K\sin hky}{k\left(1-K\varepsilon+Mk^2\right)\sin hkh-K\cos hkh}dk$$

$$-\frac{\cos\sigma t\,A(r,k_0)A(b,k_0)\cos hk_0(h-u)\cos hk_0(h-y)}{J_1^2(ak)+Y_1^2(ak_0)}$$

$$\cdot\frac{k_0\left(1-K\varepsilon+Mk_0^2\right)}{\left(1-K\varepsilon+3Mk_0^2\right)\sin h2k_0h+2k_0h\left(1-K\varepsilon+Mk_0^2\right)},\quad 0<y<u. \qquad \text{..... (4.3)}$$

Expressing the Bessel functions of the first and second kinds in terms of the Hankel functions in (4.3) we obtain

$$\varphi(r,y;t) \sim \frac{\sin\sigma t}{8\pi}\int_0^\infty \{B(r,k)+C(r,k)\}\cdot\frac{\cosh k(h-u)\left\{k\left(1-K\varepsilon+Mk^2\right)\cos hky-K\sin khy\right\}}{k\left(1-K\varepsilon+Mk^2\right)\sin hkh-K\cos hkh}dk$$

$$-\frac{\cos\sigma t}{4}\{B(r,k_0)+C(r,k_0)\}\frac{k_0\left(1-K\varepsilon+Mk_0^2\right)\cos hk_0(h-u)\cos hk_0(h-y)}{\left(1-K\varepsilon+3Mk_0^2\right)\sin h2k_0h+2k\,h\left(1-K\varepsilon+Mk_0^2\right)}$$

$$0<y<u, \qquad \text{..... (4.4)}$$

where $B(r,k)\equiv H_0^{(1)}(rk)\left\{H_0^{(2)}(bk)-\dfrac{H_0^{(1)}(bk)H_1^{(2)}(ak)}{H_1^{(1)}(ak)}\right\}$

and $C(r,k)\equiv H_0^{(2)}(rk)\left\{H_0^{(1)}(bk)-\dfrac{H_0^{(2)}(bk)H_1^{(1)}(ak)}{H_1^{(2)}(ak)}\right\}.$

We now simplify (4.4) for $r\geq b$. The case $r<b$ can be dealt with similarly. For $r\geq b$, we rotate the contour involving $C(r,k)$ clockwise by π and noting that $C\left(r,e^{-i\pi}k\right)=-B(r,k)$. Hence (4.4) can be written as

$$\varphi(r,y;t) \sim \frac{\sin\sigma t}{8\pi}\int_{-\infty}^\infty \frac{B(r,k)\cos hk(h-u)\left\{k\left(1-K\varepsilon+Mk^2\right)\cos hky-k\sin hky\right\}}{k\left(1-K\varepsilon+Mk^2\right)\sin hkh-K\cos hkh}dk$$

$$-\frac{\cos\sigma t}{4}\{B(r,k_0)+C(r,k_0)\}\frac{k_0\left(1-K\varepsilon+Mk_0^2\right)\cos hk_0(h-u)\cos hk_0(h-y)}{\left(1-K\varepsilon+3Mk_0^2\right)\sin h\;2k_0h+2k_0h\left(1-K\varepsilon+Mk_0^2\right)},$$

$$0<y<u. \qquad \text{..... (4.5)}$$

The contour of the integral (4.5) can now be extended to the contour Γ, the whole real axis with indentations above the pole at $k=-k_0$ and below the pole at $k=k_0$. Then this may be evaluated by the method of residues at the poles $k=\pm k_0$ and at $k=ik_n$ $(n=1,2,\ldots)$ by enclosing it by a semicircle of large radius in the upper half of the complex k-plane.

The contribution from the poles ik_n $(n=1,2,\ldots)$ is

$$\frac{2\sin\sigma t}{\pi}\sum_{n=1}^\infty \frac{k_n\left(1-K\varepsilon-Mk_n^2\right)\cos k_n(h-u)\cos k_n(h-y)}{\left(1-K\varepsilon-3Mk_n^2\right)\sin 2k_nh+2k_nh\left(1-K\varepsilon-Mk_n^2\right)}$$

$$\cdot K_0(rk_n)I_1(ak_n)\left[\frac{I_0(bk_n)}{I_1(ak_n)}+\frac{K_0(bk_n)}{K_1(ak_n)}\right] \qquad \text{..... (4.6)}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

where I_n, K_n (n = 0, 1) are the modified Bessel functions of first and second kinds respectively. (4.6) decays exponentially for large r, since it contains a factor $K_0(rk_n)$. Hence we finally find that for large r and t,

$$\varphi(r, y; t) - \left(\frac{2k_0}{\pi r}\right)^{1/2} \frac{\left(1-K\varepsilon+Mk_0^2\right)\cos hk_0(h-u)\cos hk_0(h-y)}{\left(1-K\varepsilon+3Mk_0^2\right)\sin h2k_0h+2k_0h\left(1-K\varepsilon+Mk_0^2\right)}$$

$$\cdot\frac{J_0(bk_0)Y_1(ak_0)-J_1(ak_0)Y_0(bk_0)}{J_1^2(ak_0)+Y_1^2(ak_0)}\left\{J_1(ak_0)\sin\left(rk_0-\sigma t-\frac{\pi}{4}\right)-Y_1(ak_0)\cos\left(rk_0-\sigma t-\frac{\pi}{4}\right)\right\}$$

$$0 < y < u. \qquad \text{..... (4.7)}$$

(4.7) represents outgoing progressive waves as $r \to \infty$.

5. Conclusion : Making $h \to \infty$, the corresponding results given by Mandal and Kundu (1988) can be deduced from (4.3) by putting $T = 0$ $\left(\text{of course for } \sigma < (g/\varepsilon)^{1/2}\right)$ and the known results given by Rhodes-Robinson (1979) can also be obtained from (4.3) by putting $\varepsilon = 0$ and making $h \to \infty$. The steady-state development of the potential due to a submerged time-harmonic ring source of circular frequency σ, outside an immersed coaxial circular cylinder in the presence of surface tension at the inertial surface is given by (4.7) for any frequency. Thus, for time-harmonic source strength, the steady-state development of the potential given by (4.7) demonstrates the existence of outgoing progressive waves at an inertial surface so long as $T > 0$. This fact was also remarked upon by Rhodes-Robinson (1982).

Acknowledgement : The author takes this opportunity to thank Prof. B.N. Mandal of Indian Statistical Institute, Calcutta, for his guidance in the preparation of this paper and U.G.C, New Delhi, for the award of a teacher fellowship (grant no. F-8-34/84 (D-II)).

References

Fenton, J.D. (1978) : Wave forces on vertical bodies of revolution. *J. fluid. Mech.* **85**, 241.

Hulme, A. (1981) : The potential of a horizontal ring of wave sources in a liquid with a free surface. *Proc.R. Soc. Lond.* **A.375**, 295.

Mandal, B.N. and Kundu, K. (1986) : A note on the singularities in the theory of water waves with an inertial surface. *J. Austral. Math. Soc.* Ser **B. 28**, 271.

Mandal, B.N. and Kundu, K. (1987) : Ring source potential in a fluid with an inertial surface in the presence of surface tension. *Int. J. Engng. Sci.* **25**, 1383.

Mandal, B.N. and Kundu, K. (1988) : Ring source potential outside a vertical coaxial circular cylinder in a liquid with an inertial surface. *J. Indian. Inst. Sci.* **68**, 261.

Mandal, B.N. and Kundu, K. (1989) : A note on the cylindrical wave-maker problem in a liquid with an inertial surface. *Int. J. Engng. Sci.* **27**, 393.

Rhodes-Robinson, P.F. (1971) : On forced surface waves due to a vertical wave-maker in the presence of surface tension. *Proc. Camb. Phil. Soc.* **70**, 323.

Rhodes-Robinson, P.F. (1979) : On surface waves in the presence of immersed vertical boundaries I. *Quart. J. Mech. Apple. Math.* **32**, 109.

Rhodes-Robinson, P.F. (1982) : Note on the effect of surface tension on water waves at an inertial surface. *J. Fluid. Mech.* **125**, 375.

Rhodes-Robinson, P.F. (1984) : On the generation of water waves at an inertial surface. *J. Austral. Math. Soc.* Ser B **25**, 366.

RABINDRA MAHAVIDYALAYA,
CHAMPADANGA, HOOGHLY,
WEST BENGAL,
INDIA

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc. **86**, 379-386 (1994)

s–ABSOLUTE ALMOST CONVERGENT SEQUENCE II

H. ÇAKALLI AND G. ÇANAK

(Received 9 March 1991; revision received 6 August 1993)

Abstract : Mursalen (1984) has investigated the absolutely almost conservative matrices, using the concept of absolute almost convergence due to Das. In this paper, we define the concept of s-absolute almost convergence for $0< s \leq 1$ in which when $s = 1$ it gives absolute almost convergence, and we investigate the s-absolutely almost conservative matrices.

1. Introduction : For an infinite matrix $A = (a_{nk})$ and an infinite sequence $x = (x_k)$ of complex numbers, we denote $A_x = \left((Ax)_n\right)$,

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k$$

where by the existence of Ax we mean the convergence of this last series for each $n \geq 0$. Given two sequence spaces X and Y, we say that a matrix A is of type (X, Y) if $Ax \in Y$ whenever $x \in X$.

Let m and c be the Banach spaces of bounded and convergent sequences $x = (x_k)$ with norm $\|x\| = \sup_k |x_k|$; and let us denote the space of sequences of bounded variation by v, *i.e.*

$$v = \left\{ x = (x_k) : \sum_{k=0}^{\infty} |x_k - x_{k-1}| < \infty, (x_{-1} = 0) \right\}$$

and v is a Banach space with the norm $\|(x_k)\| = \sum_{k=0}^{\infty} |x_k - x_{k-1}|$.

A sequence $x = (x_k)$ is said to be almost convergent if all of its Banach limits coincide. Lorentz (1948) has proved that x is almost convergent to a number L if and only if

$$\frac{1}{n+1} \sum_{k=p}^{p+n} x_k \to L \text{ as } n \to \infty \text{ uniformly in } p.$$

By f, we mean the set of all almost convergent sequences. L is said to be the f-limit of x, and we write $f\text{-}\lim x = L$.

In (Das and Mishra 1981), the idea of absolute almost convergence was briefly given in the following manner :

For any sequence $x = (x_k) \in m$, consider $t_{n,p}(x)$ defined by

$$t_{n,p}(x) = \frac{1}{n+1} \sum_{k=p}^{p+n} x_k \qquad (n, p \geq 0)$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Define for $p \geq 0$,

$$F_{n,p}(x) = t_{n,p}(x) - t_{n-1,p}(x) \quad (n \geq 1)$$

$$F_{0,p}(x) = t_{0,p}(x) = x_p \qquad \text{..... (1.1)}$$

A sequence $x = (x_k)$ is said to be absolutely almost convergent if,

$$\sum_{n=0}^{\infty} \left|F_{n,p}(x)\right| < \infty \text{ uniformly in } p.$$

The set of all absolutely almost convergent sequence is denoted by $\hat{1}$.

2. DEFINITIONS. *We define s-absolute almost convergence in the following manner : Given a positive real number s with $s \leq 1$. A sequence $x = (x_k)$ is said to be s-absolute almost convergent if*

$$\sum_{n=0}^{\infty} (1+n)^{2(1-s)} \left|F_{n,p}(x)\right|^s < \infty$$

converges uniformly in p. By $\hat{1}_1(s)$, we denote the set of all s-absolute almost convergent sequences. When s = 1, the definition of absolute almost convergence is obtained, that is $\hat{1}_1(s) = \hat{1}$.

In (Das, Kuttner and Nanda 1984), the definition of $\hat{1}$ is extended to a more general case, $\hat{1}(p)$ in the same way as 1 is extended to $1(p)$. In (1984), Mursaleen characterized absolutely almost conservative matrices. The author investigated s-absolutely almost conservative matrices for $s \geq 1$. (Çakalli 1988).

The object of this paper is to characterize s-absolutely almost conservative matrices for $0 < s \leq 1$.

Let Ax be defined. Then by (1.1), we get for $n, p \geq 0$

$$F_{n,p}(Ax) = \sum_{k=0}^{\infty} t(n,k,p)\, x_k$$

where

$$t(n,k,p) = \begin{cases} \dfrac{1}{n(n+1)} \displaystyle\sum_{i=1}^{n} i\left(a_{p+i,k} - a_{p+i-1,k}\right), & (n \geq 1) \\ a_{p,k} & (n = 0) \end{cases}$$

Mursaleen (1984) gave the following theorem :

THEOREM A. *Let A be an infinite matrix. Then A is absolutely almost conservative if and only if,*

(i) $\quad sup_{p,k} \left| \displaystyle\sum_{l=k}^{\infty} a_{pl} \right| < \infty$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(ii) *There exist a constant K such that for r, p = 1, 2,...*

$$\sum_{n=0}^{\infty}\left|\sum_{k=0}^{r} t(n,k,p)\right| \le K,$$

(iii) $(a_{pk}) \in \hat{1}$. *for each k, and*

(iv) $\left(\sum_{k=0}^{\infty} a_{pk}\right) \in \hat{1}$.

For A to be absolutely almost conservative

$$f-\lim Ax = a\lim_{k\to\infty} x_k + \sum_{k=0}^{\infty} x_k a_k$$

for every $x = (x_k) \in v$, where for each k, (a_{pk}) is almost convergent with limit a_k and $\sum_{k=0}^{\infty} a_{pk}$ is almost convergent with limit a.

Now we prove the following theorem :

THEOREM **B.** *Let $A = (a_{nk})$ be an infinite matrix. Then A is s-absolutely almost conservative if and only if*

$$sup_{p,k}\left|\sum_{l=k}^{\infty} a_{pl}\right| < \infty, \qquad \text{..... (1)}$$

there exist a constant K such that for r, p = 1,2,...

$$\sum_{n=0}^{\infty}(1+n)^{2(1-s)}\left|\sum_{k=0}^{r} t(n,k,p)\right|^{s} \le K, \qquad \text{..... (2)}$$

$(a_{pk}) \in \hat{1}_1(s)$ *for each k, and* ..... (3)

$$\left(\sum_{k=0}^{\infty} a_{pk}\right) \in \hat{1}_1(s). \qquad \text{..... (4)}$$

For A to be s-absolutely almost conservative

$$f-\lim Ax = a\lim_{k\to\infty} x_k + \sum_{k=0}^{\infty} x_k a_k$$

for every $x = (x_k) \in v$, where, for each k, (a_{pk}) is almost convergent with limit a_k and $\sum_{k=0}^{\infty} a_{pk}$ is almost convergent with limit a.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. Necessity : If $x=(x_k)= \in v$, then

$$\sum_{n=0}^{\infty}(1+n)^{2(1-s)}\left|F_{n,p}(Ax)\right|^s<\infty \text{ uniformly in } p.$$

Hence $\sum_{n=0}^{\infty}\left|F_{n,p}(Ax)\right|^s<\infty$ uniformly in p. Then, there exists a positive integer n_0 such that $\left|F_{n,p}(Ax)\right|^s<1$ for $n \geq 0$ uniformly in p. Since s is between 0 and 1 we get

$$\left|F_{n,p}(Ax)\right|<\left|F_{n,p}(Ax)\right|^s<1 \text{ uniformly in } p.$$

Hence it follows that,

$$\sum_{n=0}^{\infty}\left|F_{n,p}(Ax)\right|$$

converges uniformly in p. Then $A\in\left(v,\hat{1}_1(1)\right)$, therefore A is type (v, m). Hence condition (1) holds. Necessity of (3) and (4) is obvious since $a_k \in v$,, $e\in v$ where e_k = (0, 0,...0, 1, 0,...) here 1 appears in k-th position and e = (1, 1,...1,...). Consider, for fixed p and j,

$$T_{p,j}(x)=\sum_{k=0}^{j}a_{pk}x_k.$$

For fixed p and j, $T_{p,j}$ is a continuous linear functional on v. By the assumption, for all $x \in v$, $T_{p,j}(x)$ tends to a limit as $j\to\infty$ (p fixed) and hence, by the Banach Steinhaus theorem (Maddox 1970), this limit $(Ax)_p$ is also a continuous linear functional on v. Hence, for fixed p and fixed (finite) n,

$$\sum_{n=0}^{\infty}(1+r)^{2(1-s)}\left|F_{r,p}(Ax)\right|^s \qquad \text{..... (5)}$$

is a continuous s-norm on v. For any given $x \in v$, (5) is bounded in n, p; hence by another application of the Banach Steinhaus theorem, there exists a constant $K > 0$ such that,

$$\sum_{n=0}^{\infty}(1+n)^{2(1-s)}\left|F_{n,p}(Ax)\right|^s \leq K\|x\|. \qquad \text{..... (6)}$$

Define a sequence $x = (x_k)$ by $x_k\begin{cases}0 & ,k>r\\ 1 & ,k\leq r\end{cases}$

and applying this with (6), (2) holds.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Sufficiency. Suppose that the condition (1) – (4) hold. Take $x = (x_k) \in v$. By (4) we obtain that

$$\sum_{k=0}^{\infty} t(n,k,p)$$

converges for all n, p. Write $h(n,k,p) = \sum_{l=k}^{\infty} t(n,l,p)$ for each $k \geq 1$. For fixed n, p

$$\lim_{k\to\infty} h(n,k,p) = 0 \qquad \text{..... (7)}$$

Now by the condition (4),

$$\sum_{n=0}^{\infty} (1+n)^{2(1-s)} \left|h(n,o,p)\right|^s$$

converges uniformly in p. By condition (3), for fixed k

$$\sum_{n=0}^{\infty} (1+n)^{2(1-s)} \left|t(n,k,p)\right|^s$$

converges uniformly in p. Since

$$h(n,k,p) = h(n,o,p) - \sum_{l=0}^{k-1} t(n,l,p),$$

it follows that, for fixed k,

$$\sum_{n=0}^{\infty} (1+n)^{2(1-s)} \left|h(n,k,p)\right|^s \qquad \text{..... (8)}$$

converges uniformly in p. By (7) and the boundedness of sequence (x_k),

$$F_{n,p}(Ax) = \sum_{k=0}^{\infty} t(n,k,p)\, x_k$$

$$= \sum_{k=0}^{\infty} \left(h(n,k,p) - h(n,k+1,p)\right) x_k$$

and

$$F_{n,p}(Ax) = \lim_{m\to\infty} \sum_{k=0}^{m} h(n,k,p)(x_k - x_{k-1}) - \lim_{m\to\infty} h(n,m+1,p)\, x_m$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$= \sum_{k=0}^{\infty} h(n,k,p)(x_k - x_{k-1}).$$

We can make

$$\sum_{k=k_0+1}^{\infty} |x_k - x_{k-1}|$$

arbitrarily small by choosing k_0 sufficiently large. Hence it follows that, given $\epsilon > 0$, we can choose k_0 so that, for all p,

$$\sum_{n=0}^{\infty} (1+n)^{2(1-s)} \left| \sum_{k=k_0+1}^{\infty} h(n,k,p)(x_k - x_{k-1}) \right|^s < \frac{\epsilon}{z}.$$

By the uniform convergence of (8), it follows that once k_0 has been choosen, we can choose n_0 so that for all p,

$$\sum_{n=n_0+1}^{\infty} (1+n)^{2(1-s)} \left| \sum_{k=0}^{k_0} h(n,k,p)(x_k - x_{k-1}) \right|^s < \frac{\epsilon}{2}$$

Hence for all p,

$$\sum_{n=n_0+1}^{\infty} (1+n)^{2(1-s)} \left| \sum_{k=0}^{\infty} h(n,k,p)(x_k - x_{k-1}) \right|^s$$

$$\leq \sum_{n=n_0+1}^{\infty} (1+n)^{2(1-s)} \left| \sum_{k=0}^{\infty} h(n,k,p)(x_k - x_{k-1}) \right|$$

$$+ \sum_{n=n_0+1}^{\infty} (1+n)^{2(1-s)} \left| \sum_{k=k_0+1}^{\infty} h(n,k,p)(x_k - x_{k-1}) \right|$$

Hence

$$\sum_{n=n_0+1}^{\infty} (1+n)^{2(1-s)} \left| F_{n,p}(Ax) \right|^s < \epsilon.$$

Thus

$$\sum_{n=0}^{\infty} (1+n)^{2(1-s)} \left| F_{n,p}(Ax) \right|^s$$

converges uniformly in p. So we have obtained that $A \in \left(v, \hat{1}_1(s)\right)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

If $A \in \left(v, \hat{1}_1(s)\right)$, then $A \in \left(v, \hat{1}_1(1)\right)$. By the theorem in (Çakalli 1988), we obtain that for A to be s-absolutely almost conservative

$$f - lim\, Ax = a \lim_{k\to\infty} x_k + \sum_{k=0}^{\infty} x_k a_k$$

for every $x = (x_k) \in v$, where for each k, (a_{pk}) is almost convergent with limit a_k and $\sum_{k=0}^{\infty} a_{pk}$ is almost convergent with limit a.

Now combining the theorem in (Çokalli 1988), for $s > 0$ we get the following :

COROLLARY. *A is s-absolutely almost conservative if and only if*

$$sup_{p,k}\left|\sum_{l=k}^{\infty} a_{pl}\right| < \infty \qquad \text{..... (i)}$$

there exist a constant K such that for $r, p = 1,2,\ldots$

$$\sum_{n=0}^{\infty}(1+n)^{2|s-1|}\sum_{k=0}^{r} t(n,k,p) \le K \qquad \text{..... (ii)}$$

$$\left(a_{pk}\right) \in \hat{1}_1(s) \text{ for each } k, \text{ and} \qquad \text{..... (iii)}$$

$$\left(\sum_{k=0}^{\infty} a_{pk}\right) \in \hat{1}_1(s) \qquad \text{..... (iv)}$$

For A to be s-absolutely almost conservative,

$$f - lim\, Ax = a \lim_{k\to\infty} x_k + \sum_{k=0}^{\infty} x_k a_k$$

for every $x = (x_k) \in v$, where, for each k, (a_{pk}) is almost convergent with limit a_k and $\sum_{k=0}^{\infty} a_{pk}$ is almost convergent with limit a.

We would like to thank the referee for the suggestions.

References

Çakalli, H. (1988) : s-absolute Almost Convergent Sequences, *Pure and Applied Mathematika Sciences*, **27**, 1-2, 13, March.

Das, G., Kuttner B. and Nanda, S. (1984) : Some Sequence Spaces and Absolute Almost Convergence, *Trans.Amer.Math. Soc.* 283, 2, 729,

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Das G. and Mishra, S.K. (1981) : A Note on a theorem of Maddox on Strong Almost Convergence, *Proc. Camb. Phill. Soc.* **89**, 393.

Lorentz, G.G (1948) : A Contribution to the theory of divergent sequences, *Acta Math.* **8**, 167

Mursaleen (1984) : Absolute Almost Convergent Sequence, *Houston Journal of Math.* **10**, 427

Maddox, I.J. (1970) : *Elements of functional Analysis*, Cambr. Univ. Press,

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCES,
ISTANBUL UNIVERSITY,
VEZNECILER, 34459
ISTANBUL, TURKEY

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc. **86**, 387-394 (1994)

A GENERALISATION OF THE GENERATING FUNCTIONS OF BOAS AND BUCK

M. BHATTACHARJYA AND J. RATH

(*Received 4 October 1991*)

Abstract : In the present paper we generalise the class of generating functions studied by Boas and Buck (1956) and those studied by Rainville (1960). We also established theorems similar to those derived by them.

1. Introduction : In 1956 Boas and Buck (1956) studied an extensive class of generating functions of polynomial sets. One of their main results is that, the necessary and sufficient condition for the polynomials $p_n(x)$ to have a generating function of the form

$$A(t)\Psi(xH(t)) = \sum_{n=0}^{\infty} p_n(x)\, t^n$$

is that sequences of numbers α_k and β_k exist such that for $n \geq 1$

$$xp_n'(x) - np_n(x) = -\sum_{k=0}^{n-1} \alpha_k p_{n-1-k}(x) - x\sum_{k=0}^{n-1} \beta_k p_{n-1-k}'(x)$$

where $\Psi(t) = \sum_{n=0}^{\infty} \gamma_n t^n, \qquad \gamma_0 \neq 0$ ..... (1.1)

$$A(t) = \sum_{n=0}^{\infty} a_n t^n, \qquad a_0 \neq 0 \qquad \text{..... (1.2)}$$

$$H(t) = \sum_{n=0}^{\infty} h_n t^{n+1}, \qquad h_0 \neq 0 \qquad \text{..... (1.3)}$$

They also studied a theorem concerning the degree of $p_n(x)$. Rainville (1960) further generalised the above result by considering the polynomials $f_n(x)$ satisfying the generating relation

$$A(t)\Psi(xH(t) + g(t)) = \sum_{n=0}^{\infty} f_n(x) t^n$$

where $g(t)$ is permitted to be identically zero. In the present paper we further generalise the results of Boas and Buck (1956) as well as those of Rainville (1960) by considering the polynomial sets satisfying the generating relations

$$x^{\nu} A(t)\Psi(x^m H(t)) = \sum_{n=0}^{\infty} p(n; m, \nu; x)\, t^n \qquad \text{..... (1.4)}$$

and

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$x^{v}A(t)\Psi\left(x^{m}H(t)+g(t)\right)=\sum_{k=0}^{\infty}f(n;m,v;x)t^{n} \qquad \text{..... (1.5)}$$

where m is any positive integer any v is a non negative integer. Theorems similar to those derived by Boas and Buck (1956) and Rainville (1960) are also established.

Dutta and Manocha (1983) introduced a class of orthogonal polynomials $F_n^m(x;\lambda,v)$ of a new type. The generating functions of this class, as derived by the authors, are of type (1.4).

2. THEOREM (1). *If $p(n;m,v,x)$ is defined by (1.4) with (1.1), (1.2), and (1.3) holding, $p(n;m;v;x)$ is a polynomial in x and $p(n;m;v;x)$ is of degree precisely $mn+v$ if and only if $\gamma_n \neq 0$.*

Proof. Let us put

$$p(n;m,v;x)=\sum_{k=0}^{\infty}s(k,n)x^{k}$$

Then we have by (1.4)

$$x^{v}A(t)\Psi\left(x^{m}H(t)\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}s(k,n)x^{k}t^{n}$$

or,
$$A(t)\Psi\left(x^{m}H(t)\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}s(k,n)\,x^{k-v}t^{n}$$

Now replacing x^m by x we have

$$A(t)\Psi\left(x\,H(t)\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}s(k,n)x^{\frac{k-v}{m}}t^{n}$$

Thus r differentiations with respect to x followed by our putting $x=0$ yield

$$A(t)\left[H(t)\right]^{r}\Psi^{r}(0)=\sum_{n=0}^{\infty}r!\,s(mr+v,n)t^{n} \qquad \text{..... (2.1)}$$

Because of (1.1), (1.2) and (1.3) we have

$$A(t)\left[H(t)\right]^{r}\Psi^{r}(0)=a_0h_0^{r}r!\gamma_r t^{r}+\sum_{n=r+1}^{\infty}C(r,n)\,t^{n} \qquad \text{..... (2.2)}$$

where $C(r,n)$ is not important to us. Comparing (2.1) and (2.2) we get

$$s(mr+v,n)=0 \quad \text{for } n<r \qquad \text{..... (2.3)}$$

and

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$s(mr+\nu, r) = a_0 \quad h_0^r \gamma_r \qquad \text{..... (2.4)}$$

The condition (2.3) shows that $p\ (n;\ m,\ \nu;\ x)$ is a polynomial of degree $\leq mn + \nu$. The condition (2.4) with r replaced by n shows that $p(n;\ m,\ \nu;\ x)$ is of degree $mn + \nu$ if and only x if $\gamma_n \neq 0$, since $a_0 h_0 \neq 0$ by (1.2) and (1.3).

THEOREM (2). *For the polynomials $p(n;\ m,\ \nu;\ x)$ defined by (1.4) with (1.1), (1.2), (1.3) holding and $\gamma_n \neq 0$, there exist sequences of numbers α_k and β_k such that, for $n \geq 1$*

$$xDp(n;m,\nu;x)-(\nu+nm)p(n,m,\nu;x) =$$

$$\sum_{k=0}^{n-1}(\nu\beta_k - m\alpha_k)p(n-k-1;m,\nu;x) - \sum_{k=0}^{n-1} x\beta_k Dp(n-k-1;m,\nu;x) \qquad \text{..... (2.5)}$$

where $D = \dfrac{d}{dx}$ *and*

$$\frac{tA'(t)}{A(t)} = \sum_{n=0}^{\infty} \alpha_n t^{n+1} \qquad \text{..... (2.6)}$$

$$\frac{tH'(t)}{H(t)} = 1 + \sum_{n=0}^{\infty} \beta_n t^{n+1} \qquad \text{..... (2.7)}$$

Proof. Let us put $\quad F = x^\nu A(t)\Psi\left(x^m H(t)\right) \qquad$ ..... (2.8)

Then

$$\frac{dF}{dx} = \nu x^{\nu-1}A(t)\Psi\left(x^m H(t)\right) + x^\nu A(t) m x^{m-1} H(t)\Psi'$$

$$= \nu x^{\nu-1}A(t)\Psi\left(x^m H(t)\right) + m x^{\nu+m-1}A(t)H(t)\Psi' \qquad \text{..... (2.9)}$$

and

$$\frac{dF}{dt} = x^\nu A'(t)\Psi + x^{m+\nu}A(t)H'(t)\Psi'$$

Eliminating Ψ with Ψ' with the help of (2.8),(2.9) and (2.10)

$$t\frac{H'(t)}{H(t)}x\frac{\partial F}{\partial x} - \nu t\frac{H'(t)}{H(t)}F = mt\frac{\partial F}{\partial t} - mt\frac{A'(t)}{A(t)}F \qquad \text{..... (2.11)}$$

If α_k and β_k are given by (2.6) and (2.7) then with $F = \sum_{n=0}^{\infty} p(n;m,\nu;x)\,t^n$ we obtain from equation (2.11)

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\left[1+\sum_{k=0}^{\infty}\beta_k t^{k+1}\right]\left[\sum_{n=0}^{\infty} xDp(n;m,v;x)t^n\right]-v\left[1+\sum_{k=0}^{\infty}\beta_k t^{k+1}\right]\left[\sum_{n=0}^{\infty} p(n;m,v;x)t^n\right]$$

$$= \quad m\sum_{n=0}^{\infty} np(n;m,v;x)\,t^n - m\sum_{k=0}^{\infty}\alpha_k t^{k+1}\sum_{n=0}^{\infty} p(n;m,v;x)\,t^n$$

$$\text{or,} \quad \sum_{n=0}^{\infty}\{xDp(n;m,v;x)-vp(n;m,v;x)-mnp(n;m,v;x)\}t^n$$

$$= \quad \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}(v\beta_k - m\alpha_k)p(n-k-1;m,v;x)t^n - \sum_{n=1}^{\infty}\sum_{k=0}^{n=1} x\beta_k Dp(n-k-1;m,v;x)\,t^n$$

from which we obtain the recurrence relation

$$xDp(n;m,v;x)-vp(n;m,v;x)-mnp(n;m,v;x)$$

$$= \quad \sum_{k=0}^{n-1}(v\beta_k - m\alpha_k)p(n-k-1;m,v;x) - \sum_{k=0}^{n-1} x\beta_k Dp(n-k-1;m,v;x) \qquad \text{..... (2.12)}$$

3. An extension. Here we give some extension of the above theorems. Let us consider the generating relation

$$x^v A(t)\Psi\left(x^m H(t)+G(t)\right)=\sum_{n=0}^{\infty} f(n;m,v;x)t^n \qquad \text{..... (3.1)}$$

in which

$$\Psi(t)=\sum_{n=0}^{\infty}\gamma_n t^n, \qquad \gamma_0 \neq 0 \qquad \text{..... (3.2)}$$

$$A(t)=\sum_{n=0}^{\infty} a_n t^n, \qquad a_0 \neq 0 \qquad \text{..... (3.3)}$$

$$H(t)=\sum_{n=0}^{\infty} h_n t^{n+1}, \qquad h_0 \neq 0 \qquad \text{..... (3.4)}$$

$$G(t)=\sum_{n=0}^{\infty} g_n t^{n+2}, \qquad \text{..... (3.5)}$$

where $G(t)$ may be identically zero.

Theorem 3. *If $f(n;m,v;x)$ is defined by (3.1) with (3.2), (3.3), (3.4) and (3.5) holding $f(n;m,v;x)$ is a polynomial in x and $f(n;m,v;x)$ is of degree precisely n if and only if $\gamma_n \neq 0$.*

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. Let us write $$f(n;m,v;x)=\sum_{k=0}^{\infty}s(k,n)x^k$$

Then $$x^v A(t)\Psi\{x^m H(t)+G(t)\}=\sum_{n,k=0}^{\infty}s(k,n)x^k t^n \qquad \text{..... (3.6)}$$

Replacing x^m by x in (3.6) after multiplying both sides by x^{-v} we get

$$A(t)\Psi\{xH(t)+G(t)\}=\sum_{n,k=0}^{\infty}s(k,n)\,x^{\frac{k-v}{m}}\,t^n$$

Differentiating both sides r times and putting $x=0$ we obtain

$$A(t)[H(t)]^r\Psi^{(r)}(G(t))=\sum_{n=0}^{\infty}r!\,s(mr+v,n)t^n \qquad \text{..... (3.7)}$$

But from (3.2) (3.3), (3.4) and (3.5) we obtain

$$A(t)[H(t)]^r\Psi^{(r)}(G(t))=a_0h_0^r r!\,\gamma_r t^r+\sum_{n=r+1}^{\infty}C(r,n)\,t^n \qquad \text{..... (3.8)}$$

where $C(r,n)$ is not important for us. Comparing (3.7) and (3.8) we obtain

$$s(mr+v,n)=0 \quad \text{if } n<r \qquad \text{..... (3.9)}$$

and $$s(mr+v,r)=a_0h_0^r\gamma_r \qquad \text{..... (3.10)}$$

From (3.9) and (3.10) we conclude that $f(n;m,v;x)$ is a polynomial in x of degree precisely $mn+v$ if and only if $\gamma_n \neq 0$

THEOREM 4. *For the polynomials $f(n;m,v;x)$ defined by (3.1) with (3.2), (3.3), (3.4) and (3.5) holding and $\gamma_n \neq 0$, there exist sequences of numbers α_k, β_k and δ_k such that for $n\geq 1$*

$$xDf(n;m,v;x)-vf(n;m,v;x)-mn(n;m,v;x)$$

$$=-\sum_{k=0}^{n-1}(x\beta_k+\delta_k)Df(n-k-1;m,v;x)+\sum_{k=0}^{n-1}\left(v\beta_k+vx^{-1}\delta_k-m\alpha_k\right)f(n-k-1;m,v;x)$$

where $$\frac{tA'(t)}{A(t)}=\sum_{n=0}^{\infty}\alpha_n t^{n+1} \qquad \text{..... (3.12)}$$

$$\frac{tH'(t)}{H(t)}=1+\sum_{n=0}^{\infty}\beta_n t^{n+1} \qquad \text{..... (3.13)}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\frac{tG'(t)}{H(t)} = \sum_{n=0}^{\infty} \delta_n t^{n+1} \qquad \text{..... (3.14)}$$

Proof. Let us write

$$F = x^{\nu} A(t) \Psi\left(x^m H(t) + G(t)\right) \qquad \text{..... (3.15)}$$

Then

$$\frac{\partial F}{\partial x} = \nu x^{\nu-1} A(t)\Psi + m x^{m+\nu-1} A(t) H(t) \Psi' \qquad \text{..... (3.16)}$$

and

$$\frac{\partial F}{\partial t} = x^{\nu} A'(t)\Psi + x^{\nu} A(t)\left(x^m H'(t) + G'(t)\right)\Psi' \qquad \text{..... (3.17)}$$

Eliminating Ψ and Ψ' from (3.15), (3.16) and (3.17) we have

$$xt\frac{H'(t)}{H(t)}\frac{\partial F}{\partial x} + t\frac{G'(t)}{H(t)}\frac{\partial F}{\partial x} - \nu t\frac{H'(t)}{H(t)}F - \nu x^{-1} t\frac{G'(t)}{H(t)}F = mt\frac{\partial F}{\partial t} - mt\frac{A'(t)}{A(t)}F$$

$$= \quad m\frac{\partial F}{\partial t} - mt\frac{A'(t)}{A(t)}F$$

Substituting the values form (3.12), (3.13) and (3.14) we get

$$x\left(1 + \sum_{n=0}^{\infty}\beta_n t^{n+1}\right)\sum_{n=0}^{\infty} Df(n;m,\nu;x)\,t^n + \sum_{n=0}^{\infty}\delta_n t^{n+1}\sum_{n=0}^{\infty} Df(n;m,\nu;x)t^n$$

$$-\nu\left(1 + \sum_{n=0}^{\infty}\beta_n t^{n+1}\right)\sum_{n=0}^{\infty} f(n;m,\nu;x)\,t^n - \nu x^{-1}\sum_{n=0}^{\infty}\delta_n t^{n+1}\sum_{n=0}^{\infty} f(n;m,\nu;x)\,t^n$$

$$= \quad m\sum_{n=0}^{\infty} nf(n;m,\nu;x)\,t^n - m\sum_{n=0}^{\infty}\alpha_k t^{n+1}\sum_{n=0}^{\infty} f(n;m,\nu;x)t^n$$

$$\Rightarrow \quad \sum_{n=0}^{\infty}\left\{xDf(n;m,\nu;x) - \nu f(n;m,\nu;x) - mnf(n;m,\nu;x)\right\}t^n$$

$$= \quad \sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\left\{-x\beta_k Df(n-k-1;m,\nu;x) - \partial_k Df(n-k-1;m,\nu;x)\right\}t^n$$

$$+\sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\left\{\nu\beta_k f(n-k-1;m,\nu;x) + \nu x^{-1}\delta_k f(n-k-1;m,\nu;x) - m\alpha_k f(n-k-1;m,\nu;x)\right\}t^n$$

From the above we easily get the recurrence relation

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$xDf(n,m,v;x)-(v+mn)f(n;m,v;x)$$

(3.14) $$= \sum_{k=0}^{n-1}(x\beta_k+\delta_k)Df(n-k-1;m,v;x)+\sum_{k=0}^{n-1}\left(v\beta_k+vx^{-1}\delta_k-m\alpha_k\right)f(n-k-1;m,v;x) \quad \text{..... (3.18)}$$

4. Special cases. The generating relation (1.4) reduces to that studied by Boas and Buck (1956) for $m = 1$ and $v = 0$. Also, the relation (1.5) reduces to the generating relation studied by Rainville (1960) for $m = 1$, $v = 0$.

(3.15) Dutta and Manocha (1983) obtained the functions $F^m_{mn+v}(x;\lambda,v)$ which satisfy the generating relation

(3.16) $$\sum_{n=0}^{\infty}F^m_{mn+v}(x;\lambda,v)=x^v e_0^t F_1\left[\frac{2v+\lambda+m-1}{m};\ -x^m t\right]$$

Comparing with (1.4) we note that here $A(t) = et$ and $H(t) = -t$

(3.17) From (2.6) and (2.7) we easily get $\alpha_0 = 1$, $\alpha_n = 0$ for $n \geq 1$ and $\beta_k = 0$ for any k. With these values of α_k and β_k (2.5) reduces to

$$xDF^m_{mn+v}(x;\lambda,v)-(v+nm)F^m_{mn+v}(x;\lambda,v)=-nmF^m_{m(n-1)+v}(x;\lambda,v) \quad \text{..... (4.1)}$$

which was obtained by Dutta and Manocha (1983). The above polynomial of Dutta and Manocha also satisfies the generating relation

$$\sum_{n=0}^{\infty}\left(\frac{2v+\lambda+m-1}{m}\right)_n F^m_{mn+v}(x;\lambda,v)\frac{t^n}{n!}$$

$$= x^v(1-t)^{-\left(\frac{2v+\lambda+m-1}{m}\right)}exp\left(\frac{-x^m t}{1-t}\right)$$

Comparing with (1.4), from (2.6) and (2.7) we obtain

$\alpha_k=\left(\frac{2v+\lambda+m-1}{m}\right)$ for any k and $\beta_k = 1$ for any k. With the above values of α_k and β_k the recurrence relation (2.5) reduces to

$$xDF^m_{mn+v}(x;\lambda,v)-(mn+v)F^m_{mn+v}(x;\lambda,v)$$

$$= -\sum_{k=0}^{n-1}\frac{(v+\lambda+m-1)(-n)_{k+1}}{\left(-\frac{2v+\lambda-1+mn}{m}\right)_{k+1}}F^m_{m(n-k-1)+v}(x;\lambda,v)$$

$$-\sum_{k=0}^{n-1}\frac{x(-n)_{k+1}}{\left(-\frac{2v+\lambda-1+mn}{m}\right)_{k+1}}DF^m_{m(n-k-1)+v}(x;\lambda,v) \quad \text{..... (4.2)}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

This relation is not found in the works of Dutta and Manocha (1983).

References

Boas, R.P., Jr., and Buck, R.C. (1956) : Polynomials defined by generating relations, *Amer. Math. Monthly*, **63**, 626.

Dutta and Manocha (1983) : A class of orthogonal polynomials of a new type, *Internat. J. Math. Sci.*, **6**, 1, 171.

Rainville, E.D. (1960) : *Special functions*, Macmillan Company, New York,

DEMARTMENT OF MATHEMATICS,
KHALLIKOTE COLLEGE,
BERHAMPUR (GM.), ORISSA

Present Addredd :
DEPARTMENT OF MATHEMATICS
F. M. COLLEGE, P.O. BALASORE,
ORISSA

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc. **86**, 395-404 (1994)

θ-PERFECT FUNCTIONS BETWEEN BITOPOLOGICAL SPACES

S. K. SEN AND M. N. MUKHERJEE

(*Received 12 October 1991*)

Abstract : The paper introduces the concept of θ-perfect functions between bitopological spaces, a number of theorems, characterizing such a class of functions, show how it can be studied from different standpoints. Finally, certain applications of the concept including one yielding the characterizations of the important class of bitopological *QHC* spaces, are achieved.

Perfect functions between topological spaces is a fairly old concept which has been studied by many mathematicians. Calling it a proper map, Bourbaki (1966) gave several characterizations of it. A different approach in terms of directed families of sets was posed by Whyburn (1965). Similar investigations have resulted in the definitions and study of θ-perfect maps by Dickman and Porter (1977), D'Aristotile (1974) and others. In this paper, we aspire to extend the idea of θ-perfect maps to bitopological spaces. Several approaches to the formulation of such a class of functions have been adopted, and as a final step, it is shown that such functions can effectively be applied to the characterizations of certain important class of spaces, e.g. pairwise *QHC* spaces.

Thoughout the paper, by X and Y we shall mean bitopological spaces (X, Q_1, Q_2) and (Y, P_1, P_2) respectively. For $A \subset (X, Q_1, Q_2)$, Q_i-int and $Q_i - clA$ will respectively stand for the interior and closure of A in (X, Q_i), where $i = 1, 2$. A point x in X is called an ij-θ-contact point of a subject A of X iff for any Q_i-open neighbourhood (henceforth *nbd*, for short) U of x, $(Q_j - clU) \cap A \neq \phi$ (Kariofilli's 1986). The set of all ij-θ-contact points of A is called the ij-θ-closure of A and is denoted by ij-θ--clA. $A(\subset X)$ is called ij-θ-closed iff $A = ij$-θ-clA. In these definitions and also in what follows, whenever i and j both appear, it is to be understood that $i, j \in \{1, 2\}$ and $i \neq j$. A space (X, Q_1, Q_2) is called pairwise Hausdorff (Kelly 1963) (pairwise Urysohn (Singal and Singal 1970)) iff for any two distinct points x, y of X, there exist a Q_i-open *nbd* U of x and a Q_j-open *nbd* V of y such that $U \cap V = \phi$ (resp. $Q_j - clU \cap Q_i - clV = \phi$).

DEFINITION 1. (Mukherjee, Banerjee and Malakar 1990) *A point x in a space (X, Q_1, Q_2) is called an ij-θ-adherent point of a filterbase $\mathcal{F}$ on X iff it is an* ij-θ-*contact point of every number of $\mathcal{F}$. The set of all ij-θ-adherent points of $\mathcal{F}$ is called the ij-θ-adherence of $\mathcal{F}$ and is denoted by ij-θ-ad $\mathcal{F}$. A filterbase $\mathcal{F}$ on X is said to be ij-θ-convergent to a subset A of X, written $\mathcal{F} \xrightarrow{ij-\theta} A$ iff for each Q_i-open cover $\mathcal{U}$ of A, there is a finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$ and a member F of $\mathcal{F}$ such that $F \subset \cup\{Q_j - clU : U \in \mathcal{U}_0\}$. If $x \in X$, we say $\mathcal{F} \xrightarrow{ij-\theta} x$ iff $\mathcal{F} \xrightarrow{ij-\theta} \{x\}$ or equivalently, Q_j-closure of every Q_i-open nbd of x contains some member of $\mathcal{F}$*

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

THEOREM 1. (Mukherjee, Banerjee and Malakar 1990) *A point x of a space X is an ij-θ-adherent point of a filterbase $\mathcal{F}$ on X iff there exists a filterbase $\mathcal{F}^*$ finer than $\mathcal{F}$ such that* $\mathcal{F}^* \xrightarrow{ij-\theta} x$.

DEFINITION 2. *A filterbase $\mathcal{F}$ on (X, Q_1, Q_2) is said to be ij-θ-directed towards a subset A of X, written as $\mathcal{F} \xrightarrow{ij-\theta-d} A$, iff every filterbase $\mathcal{G}$ finer than $\mathcal{F}$ has an ij-θ-adherent point in A, i.e. $(ij-\theta-ad\mathcal{G}) \cap A \neq \phi$. We write $\mathcal{F} \xrightarrow{ij-\theta-d} x$ to mean $\mathcal{F} \xrightarrow{ij-\theta-d} \{x\}$, where $x \in X$.*

LEEMA 1. *For a filterbase $\mathcal{F}$ on X and a point $x \in X$, $\mathcal{F} \xrightarrow{ij-\theta} x$ iff $\mathcal{F} \xrightarrow{ij-\theta-d} x$.*

Proof. If $\mathcal{F}$ does not ij-θ-converge to x, then there exists a Q_i-open nbd U of x such that $F \not\subset Q_j - clU$, for all $F \in \mathcal{F}$. Then $\mathcal{G} = \{(X - Q_j - clU) \cap F : F \in \mathcal{F}\}$ is a filterbase on X finer than $\mathcal{F}$, and clearly $x \notin ij-\theta-ad\mathcal{G}$. Thus $\mathcal{F}$ cannot be *ij-θ*-directed towards x.

The converse is clear.

DEFINITION 3. *A function $f:(X, Q_1, Q_2) \to (Y, P_1, P_2)$ is said to be ij-θ-perfect iff for each filterbase $\mathcal{F}$ on f(X), ij-θ-directed towards some subset B of f(X), the filterbase $f^{-1}(\mathcal{F})$ is ij-θ-directed towards f^{-1} (B) in X.*

f is called pairwise θ-perfect iff f is 12-and 21-θ-perfect.

In the following theorem we show that only points of Y could have been sufficient for the subset B (in Definition 3) and hence *ij-θ*-direction can be replaced (in view of Lemma 1) by *ij-θ*-convergence.

THEOREM 2. *For function $f:(X, Q_1, Q_2) \to (Y, P_1, P_2)$, the following statements are equivalent :*

(a) *f is ij-θ-perfect.*

(b) *For each filterbase $\mathcal{F}$ on f(X), which is ij-θ-convergent to a point y of Y, $f^{-1}(\mathcal{F}) \xrightarrow{ij-\theta-d} f^{-1}(y)$.*

(c) *For any filterbase $\mathcal{F}$ on X,*
ij-θ-adf $(\mathcal{F}) \subset f(ij-\theta-ad\mathcal{F})$.

Proof. $(a) \Rightarrow (b)$: Follows from Lemma 1.

$(b) \Rightarrow (c)$. Let $y \in ij-\theta-adf(\mathcal{F})$. Then by Theorem 1, there is a filterbase $\mathcal{G}$ on $f(X)$ finer than $f(\mathcal{F})$ such that $\mathcal{G} \xrightarrow{ij-\theta} y$. Let $\mathcal{U} = \{f^{-1}(G) \cap F : G \in \mathcal{G} and\ F \in \mathcal{F}\}$. Then $\mathcal{U}$ is a filterbase on X finer than $f^{-1}(\mathcal{G})$. Since $\mathcal{G} \xrightarrow{ij-\theta-d} y$ (by Lemma 1) and f is *ij-θ*-perfect, $f^{-1}(\mathcal{G}) \xrightarrow{ij-\theta-d} f^{-1}(y)$. $\mathcal{U}$ being finer than $f^{-1}(\mathcal{G})$, we have $f^{-1}(y) \cap (ij-\theta-ad\mathcal{U}) \neq \phi$. It is then clear that $f^{-1}(y) \cap (ij-\theta-ad\mathcal{F}) \neq \phi$. Thus $y \in f(ij-\theta-ad\mathcal{F})$.

$(c) \Rightarrow (a)$: Let $\mathcal{F}$ be a filterbase on $f(X)$ such that it is *ij-θ*-directed towards some subset B of $f(X)$. Let $\mathcal{G}$ be a filterbase on X finer than $f^{-1}(\mathcal{G})$. Then $f(\mathcal{G})$ is a filterbase on $f(X)$ finer

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

than $\mathscr{F}$ and hence $B \cap (ij-\theta-adf(\mathscr{G})) \neq \phi$. Thus by (c), $B \cap f(ij-\theta-ad\mathscr{G}) \neq \phi$ so that $f^{-1}(B) \cap ij-\theta-ad\mathscr{G} \neq \phi$. This shows that $f^{-1}(\mathscr{F})$ is ij-θ-directed towards $f^{-1}(B)$. Hence f is ij-θ-perfect.

REMARK 1. Putting $i, j = 1$ and 2 $(i \neq j)$ in the above characterization theorem for ij-θ-perfect functions, the corresponding characterizations of pairwise θ-perfect functions can be obtained. Similar remark applies to the results that follow.

DEFINITION 4. *A function $f:(X,Q_1,Q_2) \to (Y,P_1,P_2)$ is called ij-θ-closed iff the image of each ij-θ-closed set in X is ij-θ-closed in Y.*

THEOREM 3. *A function $f: X \to Y$ is ij-θ-closed iff ij-θ-clf $(A) \subset f(ij-\theta-clA)$, for each $A \subset X$.*

Proof. Straightforward.

THEOREM 4. *An ij-θ-perfect function $f: X \to Y$ is ij-θ-closed.*

Proof. Follows from Theorem 3 and Theorem 2 $((a) \Rightarrow (c))$ by taking $\mathscr{F} = \{A\}$.

DEFINITION 5. (Bandyopadhyay 1990) *A subset A of (X,Q_1,Q_2) is said to be ij-rigid in X iff for each filterbase $\mathscr{F}$ on X with $(ij-\theta-ad\mathscr{F}) \cap A = \phi$, there is $U \in Q_i$ and $F \in \mathscr{F}$ such that $A \subset U$ and $(Q_j - clU) \cap F = \phi$, or equivalently, iff for each filterbase $\mathscr{F}$ on X whenever $A \cap ij-\theta-ad\mathscr{F} = \phi$, then for some $F \in \mathscr{F}$, $A \cap ij-\theta-clF = \phi$.*

THEOREM 5. *If a function $f:(X,Q_1,Q_2) \to (Y,P_1,P_2)$ is ij-θ-closed such that each $f^{-1}(y)(y \in Y)$ is ij-rigid in X, then f is ij-θ-perfect.*

Proof. Let $\mathscr{F}$ be a filterbase on $f(X)$ such that $\mathscr{F} \xrightarrow{ij-\theta} y$ in Y, for some $y \in Y$. If $\mathscr{G}$ is a filterbase on X finer than the filterbase $f^{-1}(\mathscr{F})$, then $f(\mathscr{G})$ is a filterbase on Y, finer than $\mathscr{F}$. Since $\mathscr{F} \xrightarrow{ij-\theta-d} y$ (by Lemma 1), $y \in ij$-θ-adf $(\mathscr{G})$, i.e., $y \in \cap\{ij-\theta-clf(G): G \in \mathscr{G}\}$ and hence $y \in \cap\{f(ij-\theta-clG): G \in \mathscr{G}\}$ (by Theorem 3, since f is ij-θ-closed). Then $f^{-1}(y) \cap ij-\theta-clG \neq \phi$, for all $G \in \mathscr{G}$. Hence for all $U \in Q_i$ with $f^{-1}(y) \subset U$, $Q_j - clU \cap G \neq \phi$, for all $G \in \mathscr{G}$. Since $f^{-1}(y)$ is ij-rigid, it then follows that $f^{-1}(y) \cap ij-\theta-ad\mathscr{G} \neq \phi$. Thus $f^{-1}(\mathscr{F}) \xrightarrow{ij-\theta-d} f^{-1}(y)$. Hence by Theorem 2 $((b) \Rightarrow (a))$, f is ij-θ-perfect.

In order to give a natural example of an ij-θ-perfect function, we recall that a subset A of a space (X,Q_1,Q_2) is called an ij-ro-set iff $A = Q_i$-int Q_j-clA (Singal and Arya 1991). The collection of all ij-ro-sets forms a base for a topology Q_i^s (say) on X, strictly weaker than Q_i (for $i, j = 1, 2$ and $i \neq j$). Denoting the *ij-θ-closure of a set A in* (X, Q_1, Q_2) $(resp.(X,Q_1^s,Q_2^s))$ by $Q_iQ_j-\theta-clA$ $(resp.Q_i^sQ_j^s-\theta-clA)$, one can prove the following lemma in a straightforward manner.

LEMMA 2. *For any subset A of a space (X,Q_1,Q_2), $Q_iQ_j-\theta-clA = Q_i^sQ_j^s-\theta-clA$.*

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

THEOREM 6. *For any space* (X, Q_1, Q_2), *the identity maps* $i_1:(X, Q_1, Q_2) \to (X, Q_1^s, Q_2^s)$, *and* $i_2:(X, Q_1^s, Q_2^s) \to (X, Q_1, Q_2)$ *are ij-θ-perfect.*

Proof. For each $x \in X$, $i_1^{-1}(x)$ and $i_2^{-1}(x)$ are obviously ij-rigids. Again, for any $A \subset X$ we have by Lemma 2, $Q_i Q_j - \theta - clA = Q_i^s Q_j^s - \theta - clA$, so that i_1 and i_2 are ij-θ-closed. Thus by Theorem 5, i_1 and i_2 are ij-θ-perfect.

DEFINITION 6. (Bose and Sinha 1981) *A function* $f: (X, Q_1, Q_2) \to (Y, P_1, P_2)$ *is called ij-θ-continuous iff for any* P_i*-open nbd V of f(x), there exists a* Q_i*-open nbd U of x such that* $f(Q_j - clU) \subset P_j - clV$.

THEOREM 7. *If an ij-θ-continuous function* $f:(X, Q_1, Q_2) \to (Y, P_1, P_2)$ *is ij-θ-perfect then f is ij-θ-closed and for each* $y \in Y$, *the point inverse* $f^{-1}(y)$ *is ij-rigid in X.*

Proof. It has already been shown in Theorem 4 that an ij-θ-perfect function is ij-θ-closed. To prove the other part, let $y \in Y$, and suppose $\mathcal{F}$ is a filterbase on X such that $(ij - \theta - ad\mathcal{F}) \cap f^{-1}(y) = \phi$. Then $y \notin f(ij - \theta - ad\mathcal{F})$. Since f is ij-θ-perfect, by Theorem 2 $((a) \Rightarrow (c))$ $y \notin ij - \theta - adf(\mathcal{F})$. Thus there exists an $F \in \mathcal{F}$ such that $y \notin ij$-θ-$clf(F)$. There exists a P_i-open nbd V of y such that $P_j - clV \cap f(F) = \phi$. Since f is ij-θ-continuous, for each $x \in f^{-1}(y)$ we shall get a Q_i-open nbd U_x of x such that $f(Q_j - clU_x) \subset P_j - clV \subset Y - f(F)$. Then $f(Q_j - clU_x) \cap f(F) = \phi$, so that $Q_j - clU_x \cap F = \phi$. Then $x \notin ij - \theta - clF$, for all $x \in f^{-1}(y)$, so that $f^{-1}(y) \cap ij - \theta - clF = \phi$. Hence $f^{-1}(y)$ is ij-rigid in X.

Combining Theorems 5 and 7 we obtain :

COROLLARY 1. *An ij-θ-continuous function* $f : X \to Y$ *is ij-θ-perfect iff f is ij-θ-closed and each point inverse* $f^{-1}(y)$ *(for* $y \in Y$*) is ij-rigid in X.*

Next we show that the above theorem remains valid if ij-θ-closedness of f is replaced by a strictly weaker condition which we shall call weak ij-θ-closedness of f. Thus we define as follows.

DEFINITION 7. *A function of* $f:(X, Q_1, Q_2) \to (Y, P_1, P_2)$, *is said to be weakly ij-θ-closed iff for every* $y \in f(X)$ *and every* Q_i*-open set U containing* $f^{-1}(y)$ *in X, there exists a* P_i*-open nbd V of y such that* $f^{-1}(P_j - clV) \subset Q_j - clU$.

LEEMA 3. (Kariofillis 1986) *In a space* (X, Q_1, Q_2), *if* $U \in Q_j$, *then ij-θ-clU = Q*$_i$*-clU.*

THEOREM 8. *An ij-θ-closed function* $f:(X, Q_1, Q_2) \to (Y, P_1, P_2)$ *is weakly ij-θ-closed.*

Proof. Let $y \in f(X)$ and let U be a Q_i-open set containing $f^{-1}(y)$ in X. Now $Q_i - cl(X - Q_j - clU) = ij - \theta - cl(X - Q_j - clU)$ (by Lemma 3) and hence $f[Q_i - cl(X - Q_j - clU)]$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$= f\left[ij-\theta-cl\left(X-Q_j-clU\right)\right] \supset ij-\theta-clf\left(X-Q_j-clU\right)$ (by Theorem 3, since f is ij-θ-closed). Now since $y \notin f\left[Q_i-cl\left(X-Q_j-clU\right)\right], y \notin ij-\theta-clf\left(X-Q_j-clU\right)$ and thus there exists a P_i-open nbd V of y in Y such that $P_j-clV \cap f\left(X-Q_j-clU\right)=\phi$ which implies that $f^{-1}\left(P_j-clV\right) \cap \left(X-Q_j-clU\right)=\phi$, i.e., $f^{-1}\left(P_j-clV\right) \subset Q_j-clU$, and thus f is weakly ij-θ-closed.

That the converse of the above theorem is not true, in general, is shown below.

EXAMPLE 1. Let $f: X \to Y$ be a constant map. Then for any topologies Q_1, Q_2, P_1 and P_2, the map $f:(X,Q_1,Q_2) \to (Y,P_1,P_2)$ is weakly ij-θ-closed, for $i, j = 1$ and 2 $(i \neq j)$. Now, let $X = Y = R$ (= the set of reals). If P_1 or P_2 is the indiscrete topology on Y, then $f:(X,Q_1,Q_2) \to (Y,P_1,P_2)$ given by $f(x) = 0$, for all $x \in X$, is neither 12-θ-closed nor 21-θ-closed, irrespective of the topologies Q_1, Q_2 and P_2 (or P_1).

THEOREM 9. *Let $f:(X,Q_1,Q_2) \to (Y,P_1,P_2)$ be ij-θ-continuous. Then f is ij-θ-perfect if*

(a) *f is weakly ij-θ-closed, and*

(b) *$f^{-1}(y)$ is ij-rigid, for each $y \in Y$.*

Proof. Suppose $f:(X,Q_1,Q_2) \to (Y,P_1,P_2)$ is an ij-θ-continuous function satisfying the conditions (a) and (b). To prove that f is ij-θ-perfect we have to show in view of Theorem 5 that f is ij-θ-closed. Let $y \in ij$-θ-clf (A), for some non-null subset A of X, but $y \notin f(ij-\theta-clA)$. Then $\mathscr{B} = \{A\}$ is a filterbase on X and $(ij-\theta-ad\mathscr{B}) \cap f^{-1}(y) = \phi$. By ij-rigidity of $f^{-1}(y)$, there is a Q_i-open set U containing $f^{-1}(y)$ such that $Q_j-clU \cap A = \phi$. By weak ij-θ-closedness of f, there exists a P_i-open nbd V of y such that $f^{-1}\left(P_j-clV\right) \subset Q_j-clU$, which implies that $f^{-1}\left(P_j-clV\right) \cap A = \phi$, i.e., $\left(P_j-clV\right) \cap f(A) = \phi$ which is impossible since $y \in ij-\theta-clf(A)$. Hence $y \in f(ij-\theta-clA)$. So f is ij-θ-closed.

Combining Theorems 8 and 9 we get :

COROLLARY 2. *Let $f: (X,Q_1,Q_2) \to (Y,P_1,P_2)$ be an ij-θ-continuous function. Then f is ij-θ-perfect iff*

(a) *f is weakly ij-θ-closed, and*

(b) *$f^{-1}(y)$ is ij-rigid, for each $y \in Y$.*

DEFINITION 8. (Mukherjee, Banerjee and Malakar 1990) *In a space (X,Q_1,Q_2), a subset A will be called an ij-H-set iff for each Q_i-open cover $\mathscr{U}$ of A, there is a finite subcollection $\mathscr{U}_0$ of $\mathscr{U}$ such that $A \subset \cup\{Q_j-clU : U \in \mathscr{U}_0\}$; A is called a pairwise H-set iff it is a 12- and a 21-H-set.*

If A is an ij-H-set (pairwise H-set) and $A = X$, then the space is called an ij-QHC (resp. pairwise QHC) space (Mukherjee 1982, 1990).

It is known (Bandyopadhyay 1990) that an ij-H-set is an ij-rigid set but not conversely.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

THEOREM 10. (Mukherjee, Banerjee and Malaker 1990) *A subset A of a space (X, Q_1, Q_2) is an ij-H-set iff for each filterbase $\mathcal{F}$ on A, $(ij-\theta-ad\mathcal{F}) \cap A \neq \phi$.*

THEOREM 11. *If $f:(X, Q_1, Q_2) \to (Y, P_1, P_2)$ is ij-θ-perfect and $B(\subset Y)$ is an ij-H-set in Y, then $f^{-1}(B)$ is an ij-H-set in X.*

Proof. Let $\mathcal{F}$ be a filterbase on $f^{-1}(B)$, then $f(\mathcal{F})$ is a filterbase on B. Since B is an ij-H-set in Y, $B \cap ij-\theta-adf(\mathcal{F}) \neq \phi$ (by Theorem 10). By theorem 2 $((a) \Rightarrow (c))$, $B \cap f(ij-\theta-ad\mathcal{F}) \neq \phi$, so that $f^{-1}(B) \cap ij-\theta-ad\mathcal{F} \neq \phi$. Hence by Theorem 10, $f^{-1}(B)$ is an ij-H-set in X.

We show as follows that the converse of the above theorem is false, in general.

EXAMPLE 2. Let $X = Y = R$ (= the set of reals), Q_1 and Q_2 be the cofinite and discrete topologies on X, and P_1, P_2 respectively denote the indiscrete and usual topologies on Y. Suppose $f:(X, Q_1, Q_2) \to (Y, P_1, P_2)$ is the identity map. Each subset of either of (X, Q_1, Q_2) and (Y, P_1, P_2) is a 12-H-set. Now, any non-void finite set $A(\subset X)$ is 12-θ-closed in X, but $f(A)$ (*i. e.*, A) is not 12-θ-closed in Y (in fact, the only 12-θ-closed subsets of Y are Y and ϕ).

The theorem 11 and the above example suggest the definition of a strictly weaker version of ij-θ-perfect functions as given below :

DEFINITION 9. *A function $f: X \to Y$ is said to be almost ij-θ-perfect iff $f^{-1}(K)$ is an ij-H-set in X for every ij-H-set K in Y.*

As is expected in analogy to Theorem 5, a sufficient condition for a function to be almost ij-θ-perfect, is proved as follows.

THEOREM 12. Let $f: X \to Y$ *be any function such that*

(a) *$f^{-1}(y)$ is ij-rigid, for all $y \in Y$, and*

(b) *f is weakly ij-θ-closed.*

then f is almost ij-θ-perfect.

Proof. Let B be an ij-H-set in Y *and let* $\mathcal{F}$ be a filterbase on $f^{-1}(B)$. Now $f(\mathcal{F})$ is a filterbase on B and so by Theorem 10, $(ij-\theta-adf(\mathcal{F})) \cap B \neq \phi$. Let $y \in [ij-\theta-adf(\mathcal{F})] \cap B$. Suppose that $\mathcal{F}$ has no ij-θ-adherent point in $f^{-1}(B)$ so that $(ij-\theta-ad\mathcal{F}) \cap f^{-1}(y) = \phi$. Since $f^{-1}(y)$ is ij-rigid, there exist an $F \in \mathcal{F}$ and a Q_i-open set U containing $f^{-1}(y)$ such that $F \cap Q_j - clU = \phi$. By weak ij-θ-closedness of f, there is a P_i-open nbd V of y such that $f^{-1}(P_j - clV) \subset Q_j - clU$ which implies that $f^{-1}(P_j - clV) \cap F = \phi$, *i.e.*, $P_j - clV \cap f(F) = \phi$, which is a contradiction. Thus by Theorem 10, $f^{-1}(B)$ is an ij-H-set in X and hence f is almost ij-θ-perfect.

In order to investigate for the conditions under which an almost ij-θ-perfect function may be ij-θ-perfect, we introduce the following definition.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEFINITION 10. A function $f:(X,Q_1,Q_2)\to(Y,P_1,P_2)$ is said to be P_jQ_i-θ-continuous iff for any P_j-open nbd V of $f(x)$, there exists a Q_j-open nbd U of x such that $f(Q_j-clU)\subset P_i-clV$.

The relevance of the above definition to the characterizations of ij-θ-perfect functions is quite apparent from the following result.

THEOREM 13. *If $f:X\to Y$ is P_jQ_i-θ-continuous and Y is pairwise Urysohn, then f is ij-θ-perfect iff for every filterbase $\mathscr{F}$ on X, if $f(\mathscr{F})\xrightarrow{ij-\theta}y(\in f(X))$ then ij-θ-ad $\mathscr{F}\neq\phi$.*

Proof. Let f be ij-θ-perfect and $f(\mathscr{F})\xrightarrow{ij-\theta}y\in f(X)$, for a filterbase $\mathscr{F}$ on X. Then $f^{-1}f(\mathscr{F})\xrightarrow{ij-\theta-d}f^{-1}(y)$. Since $\mathscr{F}$ is finer than $f^{-1}f(\mathscr{F})$, $f^{-1}(y)\cap ij-\theta-ad\mathscr{F}\neq\phi$, so that ij-θ-ad $\mathscr{F}\neq\phi$. Conversely, suppose that for every filterbase $\mathscr{F}$ on X, $f(\mathscr{F})\xrightarrow{ij-\theta}y\in f(X)$ implies ij-θ-ad$\mathscr{F}\neq\phi$. Let $\mathscr{G}$ be a filterbase on $f(X)$ such that $\mathscr{G}\xrightarrow{ij-\theta}y\in f(X)$, and suppose that $\mathscr{G}^*$ is a filterbase on X such that $\mathscr{G}^*$ is finer than $f^{-1}(\mathscr{G})$. Then $f(\mathscr{G}^*)$ is finer than $\mathscr{G}$. So $f(\mathscr{G}^*)\xrightarrow{ij-\theta}y$. Hence ij-θ-ad$\mathscr{G}^*\neq\phi$. Let $z\in Y$ such that $z\neq y$. Then since Y is pairwise Urysohn, there exist a P_i-open nbd U of y and a p_j-open nbd V of z such that $(P_j-clU)\cap(P_i-clV)=\phi$. Since $f(\mathscr{G}^*)\xrightarrow{ij-\theta}y$, there exists a $G\in\mathscr{G}^*$ such that $f(G)\subset P_j-clU$. Now, since f is P_jQ_i-θ-continuous, corresponding to each $x\in f^{-1}(z)$ there is a Q_i-open nbd W of x such that $f(Q_j-clW)\subset P_i-clV$. Thus $Q_j-clW\cap G=\phi$. It follows that $f^{-1}(z)\cap ij-\theta-ad\mathscr{G}^*=\phi$, for each $z\in Y-\{y\}$. Consequently, $f^{-1}(y)\cap ij-\theta-ad\mathscr{G}^*\neq\phi$, and hence f is ij-θ-perfect.

It now follows in view of Theorem 10 that

COROLLARY 3. *If $X\to Y$ is PjQi-θ-continuous, X is ij-QHC and Y is pairwise Urysohn, then f is ij-θ-perfect.*

DEFINITION 11. *A space (X,Q_1,Q_2) is said to be locally ij-QHC iff for every $x\in X$, there is a Qi-nbd of x, which is an ij-H-set.*

LEMMA 4. (Mukherjee, Banerjee and Malakar 1990) *In a pairwise Urysohn space (X,Q_1,Q_2), an ij-H-set is ji-θ-closed.*

THEOREM 14. *Let $f:(X,Q_1,Q_2)\to(Y,P_1,P_2)$ be a PjQi-θ-continuous function from X to a locally ij-QHC, pairwise Urysohn space Y. Then f is ij-θ-perfect iff f is almost ij-θ-perfect.*

Proof. If f is ij-θ-perfect, then by Theorem 11, is almost ij-θ-perfect. Conversely, let $\mathscr{F}$ be any filterbase on X and let $f(\mathscr{F})\xrightarrow{ij-\theta}y\in f(X)$. There are an ij-H-set B in Y and a P_i-open nbd V of y such that $y\in V\subset B$. Let $\mathscr{H}=\{P_j-clU\cap f(F)\cap B:F\in\mathscr{F}$ and U is a P_i-open nbd of $y\}$. By Lemma 4, B is ji-θ-closed and hence no member of $\mathscr{H}$ is void. In fact, if

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

not, let for some P_i-open nbd U of y and some $F \in \mathscr{F}$, $P_j - clU \cap f(F) \cap B = \phi$. Then $W = U \cap V(\ni y) \in P_i$ and P_j-clW $= ji$-θ-clW (by Lemma 3) $\subset ji - \theta - clB = B$. Now, $\phi = P_j - clW \cap f(F) \cap B = P_j - clW \cap f(F)$, which is not possible, since $f(\mathscr{F}) \xrightarrow{ij-\theta} y$. Thus $\mathscr{H}$ is filterbase on Y, and is clearly finer than $f(\mathscr{F})$, so that $\mathscr{H} \xrightarrow{ij-\theta} y$. Also $\mathscr{G} = \{f^{-1}(H) \cap F : H \in \mathscr{H} \text{ and } F \in \mathscr{F}\}$ is clearly a filterbase on $f^{-1}(B)$. Since f is almost ij-θ perfect, $f^{-1}(B)$ is an ij-H-set and hence ij-θ-ad$\mathscr{G} \cap f^{-1}(B) \neq \phi$. Thus ij-θ-$ad\,\mathscr{F} \neq \phi$. Then by Theorem 13, f is ij-θ-perfect.

We now give some applications of ij-θ-perfect maps. The following characterization theorem for an ij-θ-continuous function is recalled to this end.

THEOREM 15. (Sen 1993) A function $f : X \to Y$ is ij-θ-continuous iff $f(ij - \theta - clA) \subset ij - \theta - clf(A)$, for each $A \subset X$.

THEOREM 16. *Let $f : X \to Y$ be ij-θ-continuous and ij-θ-perfect. Then f^{-1} preserves ij-rigidity.*

Proof. Let B be any ij-rigid set in Y, and let $\mathscr{F}$ be a filterbase on X such that $f^{-1}(B) \cap ij - \theta - ad\mathscr{F} = \phi$. Since f is ij-θ-perfect and $B \cap f(ij - \theta - ad\mathscr{F}) = \phi$, by Theorem 2$((a) \Rightarrow (c))$ we get $B \cap ij - \theta - adf(\mathscr{F}) = \phi$. Now B being an ij-rigid set in Y, there exists an $F \in \mathscr{F}$ such that $B \cap ij - \theta - clf(F) = \phi$. Since f is ij-θ-continuous, by Theorem 15 it follows that $B \cap f(ij - \theta - clF) = \phi$. Thus $f^{-1}(B) \cap \underset{\circ}{ij} - \theta - clF = \phi$. This proves that $f^{-1}(B)$ is ij-rigid.

It was proved in (Sen and Nandi 1992) that a space (X, Q_1, Q_2) is pairwise Hausdorff iff $\{x\} = ij - \theta - cl\{x\}$, for each $x \in X$. It then follows immediately in view of Theorem 4 that

THEOREM 17. If $f : X \to Y$ is an ij-θ-perfect subjection with X a pairwise Hausdorff space, then Y is also pairwise Hausdorff.

Our next theorem gives a characterization of an importnat class of spaces viz. the ij-QHC spaces in terms of ij-θ-perfect functions.

THEOREM 18. *For a space (Y, P_1, P_2), the following statements are equivalent :*

(a) *Y is ij-QHC.*

(b) *The constant function $g : Y \to Z$ is ij-θ-perfect, where Z is a singleton with two equal topologies viz. the unique topology on Z.*

(c) *For all spaces (X, Q_1, Q_2) the projection map $p_1 : (X \times Y, T_1, T_2) \to (X, Q_1, Q_2)$ is ji-θ-perfect, where $T_i = Q_i \times P_j (i, j = 1, 2 \text{ and } i \neq j)$.*

Proof. $(a) \Rightarrow (b)$: g is clearly ij-θ-closed. Also, $g^{-1}(Z)$, i.e., Y is obviously ij-rigid since Y is ij-QHC. Then by Theorem 5, g is ij-θ-perfect.

$(b) \Rightarrow (a)$: Follows from Theorem 11.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$(a) \Rightarrow (c)$: We show that p_1 is ji-θ-closed and that for each $x \in X$, $p^{-1}(x)$ is ji-rigid in $X \times Y$. Then the result will follow from Theorem 5.

Let $A \subset X \times Y$ and $a \notin p_1(ji-\theta-clA)$. For each $y \in Y$, $(a,y) \notin ji-\theta-clA$, so that there exist a Q_j-open nbd G_y of a and a P_i-open nbd H_y of y such that $\left[T_i - cl\left(G_y \times H_y\right)\right] \cap A = \phi$. Since Y is ij-QHC, $\{a\} \times Y$ is a ji-H-set in $X \times Y$. Thus there exist finitely many elements $y_1, y_2, \ldots, y_n$ of Y with $\{a\} \times Y \subset \bigcup_{k=1}^{n} T_i - cl\left(G_{y_k} \times H_{y_k}\right)$. Now, $a \in \bigcap_{k=1}^{n} G_{y_k} = G$ (say) which is a Q_j-open nbd of a such that $(Q_i - clG) \cap p_1(A) = \phi$. Hence $a \notin ji$-θ-$clp_1(A)$ and thus $ji-\theta-clp_1(A) \subset p_1(ji-\theta-clA)$. So p_1 is ji-θ-closed, by Theorem 3. Next, let $a \in X$. To show $p_1^{-1}(a)$ to be ji-rigid in $X \times Y$, let $\mathcal{F}$ be a filterbase on $X \times Y$ such that $p_1^{-1}(a) \cap ji-\theta-ad\mathcal{F} = \phi$. for each $y \in Y$, $(a,y) \notin ji-\theta-ad\mathcal{F}$. Thus there exist a Q_j-open nbd U_y of a in X, a P_i-open nbd V_y of y in Y and an $F_y \in \mathcal{F}$ such that $T_i - cl\left(U_y \times V_y\right) \cap F_y = \phi$, i.e. $\left(Q_i - clU_y \times P_j - clV_y\right) \cap F_y = \phi$. As shown above, there exists finitely many elements $y_1, y_2, \ldots, y_n$ of Y such that $\{a\} \times Y \subset \bigcup_{k=1}^{n} T_i - cl\left(U_{y_k} \times V_{y_k}\right)$. Putting $U = \bigcap_{k=1}^{n} U_{y_k}$ and choosing $F \in \mathcal{F}$ with $F \subset \bigcap_{K=1}^{n} F_{y_k}$, we get $\{a\} \times Y \subset U \times Y \in T_j$ such that $T_i - cl(U \times Y) \cap F = \phi$. Thus $(ji-\theta-clF) \cap \left[p_1^{-1}(a)\right] = \phi$. Hence $p_1^{-1}(a)$ is ji-rigid in $X \times Y$.

$(c) \Rightarrow (a)$: Taking $Z = X$, we have that $p_1 : Z \times Y \to Z$ is ji-θ-perfect. Therefore by Theorem 11, $Z \times Y$ *is a* ji-H-set and hence Y is *ij-QHC*.

Acknowledgement. The first author thanks U.G.C., New Delhi, for the award of Teacher-Fellowship under the F.I.P. Scheme.

References

Bandyopadhyay, G.K. (1990) : Study of some typical problem of sete topology in generalized settings, *Ph. D. Thesis,, University of Calcutta,*

Bose S. and Sinha, D. (1981) : Almost open, almost closed, θ-continuous and almost quasi-compact mappings in bitopological spaces, *Bull. Cal. Math. Soc.* **73,** 345.

Bourbaki, N. (1966) : *General Topology,* Addison-Wesley, Reading Mass.

D'Aristotile, A.J. (1974) On θ-perfect Mappings, *Boll. Un. Mat Ital.* **9,** 655.

Dickman, R.F. and Porter, J.R. (1977) : θ–perfect and θ-absolutely closed function, *Illinois Jour. Math.* **21,** 42.

Kariofillis, C.G. (1986) : On pairwise almost compactness, *Ann. Soc. Sci. Bruxelles* **100,** 129.

Kelly, J.C. (1963) : Bitopological spaces, *Proc. London Math. Soc.* **13,** 71.

Mukherjee, M.N. (1982) : On pairwise almost compactness and pairwise *H*-closedness in a bitopological space, *Ann. Soc.* Sci. Bruxelles *96, 98.*

Mukherjee, M.N., Banerjee G.K. and Malakar, S. (1990) : Bitopological QHC spaces, *Indian Jour. Pure Appl. Math* **21(7),** 639.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Sen, S.K. (1993) : Some non-continuous functions and separation axioms in bitopological spaces, *Soochow. Jour. Math.* 19, 1

Sen S.K. and **Nandi, J.N.** (1993) : Characterizations of some bitopological separation axioms in terms of *ij*-θ-closure operator *Math-Vesnik* 44, 75.

Singal A.R. and **Arya S.P.** (1991) : On pairwise almost regular spaces, *Glasnik Mat.* (6(26), 335

Singal M.K. and **Singal A.R.** (1970) : Some more separation axioms in bitopological spaces, *Ann, Soc. Sci. Bruxelles* 84, 207

Whyburn, G.T. (1965) : Directed families of sets and closedness of functions, *Proc. Nat. Acad. Sci. U.S.A.* 54, 688.

DEMARTMENT OF PURE MATHEMATICS,
UNIVERSITY OF CALCUTTA
35, BALLYGUNGE CIRCULAR ROAD,
CALCUTTA-700 019, INDIA.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc. **86**, 405-412 (1994)

SEMI-INVARIANT SUBMANIFOLDS OF A KENMOTSU MANIFOLD

B. B. SINHA AND R. N. YADAV

(*Received 2 December 1991*)

Abstract : In the present paper we study the integrability conditions of distributions and the geometry of leaves on a semi-invariant submanifolds in a Kenmotsu manifold.

1. Introduction. Let $\overline{M}$ be a $(2n+1)$ – dimensional almost contact metric manifold with almost contact metric structure (F, ξ, η, g) where F is a tensor field of type (1, 1), ξ is a vector field, η is a l-form and g is a Riemannian metric on $\overline{M}$. These tensor fields satisfy

$$F^2 = -1 + \eta \otimes \xi; \quad F\xi = 0, \quad \eta(\xi) = 1, \quad \eta o F = 0, \qquad \text{..... (1.1)}$$

and $$g(FX, FY) = g(X, Y) - \eta(X)\eta(Y) \qquad \text{..... (1.2)}$$

for all vector fields X, Y tangent to $\overline{M}$, where I designes the identity morphism on the tangent bundle $T\overline{M}$.

Throughout the paper all manifolds and morphisms are considered to be differentiable of class C^∞.

It is well known that $\overline{M}$ is a Kenmotsu manifold if and only if we have

$$(\overline{\nabla}_X F)Y = -\eta(Y)FX - g(X, FY)\xi, \qquad \text{..... (1.3)}$$

and $$\overline{\nabla}_X \xi = X - \eta(X)\xi, \qquad \text{..... (1.4)}$$

for all X, Y tangent to $\overline{M}$, where $\overline{\nabla}$ is the Riemannian connection with respect to g.

Now let M be an m-dimensional manifold isometrically immersed in a Kenmotsu manifold $\overline{M}$. Denote by TM and $TM^\perp$ the tangent bundle of M and respectively the normal bundle to M. Suppose the structure vector field ξ of $\overline{M}$ be tangent to the submanifold M and denote by $\{\xi\}$ the 1-dimensional distribution spanned by ξ on M and by $\{\xi\}^\perp$ the complementary orthogonal distribution to $\{\xi\}$ in TM.

For any vector bundle H on M we denote by $\Gamma(H)$ the module of all differentiable sections of H.

DEFINITION 1.1. *The submanifold M of the Kenmotsu manifold $\overline{M}$ is said to be semi-invariant if it endowed with the pair of orthogonal distributions $(D, D^\perp)$ satisfying the conditions :*

(i) $TM = D \oplus D^\perp \oplus \{\xi\}$

(ii) *the distribution D is invariant by F, that is*

$$F(D_x) = D_x, \text{ for each } x \in M,$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(iii) *the distribution* $D^{\perp}$ *is anti-invariant by F, that is*

$$FD_x^{\perp} \subset T_x M^{\perp}, \text{ for each } x \in M.$$

Suppose the dimension of D_x $\left(resp. D_x^{\perp}\right)$ being $2p$ (resp. q). Then if we have $p = 0$ (resp. $q = 0$) the semi-invariant submanifold becomes an anti-invariant submanifold (resp. an invariant submanifold). A generic submanifold is also a particular case of semi-invariant submanifold characterized by the condition $q = \dim T_x M^{\perp}$, for any $x \in M$. We say that M is a proper semi-invariant submanifold if it is a semi-invariant submanifold, which is neither an invariant nor an anti-invariant submanifold.

We denote by same symbol g both metrices on $\overline{M}$ and M. The projection morphisms of TM to D and $D^{\perp}$ are denoted respectively by P and Q. For any $X \in \Gamma(TM)$ and $N \in \Gamma\left(TM^{\perp}\right)$ we have

$$X = PX + QX + \eta(X)\xi, \text{ where } \eta(X) = g(X, \xi) \qquad \text{..... (1.5)}$$

$$FN = tN + fN, \qquad \text{..... (1.6)}$$

where tN (resp. fN) is the tangential (resp. normal) part of FN.

Next, we denote by ν the orthogonal complementary vector bundle to $FD^{\perp}$ in $TM^{\perp}$. Thus we have

$$TM^{\perp} = FD^{\perp} \oplus_{\nu}$$

The equations of Gauss and Weingarten for the immersion of M in $\overline{M}$ are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \text{ and respectively} \qquad \text{..... (1.7)}$$

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \qquad \text{..... (1.8)}$$

for any $X, Y \in \Gamma(TM)$ and $N \in TM^{\perp}$, where ∇ is the Levi-civita connection on M, $\nabla^{\perp}$ is the linear connection induced by $\overline{\nabla}$ on the normal bundle $TM^{\perp}, h$ is the second fundamental form of M and A_N is the fundamental tensor of Weingarten with respect to the normal section N. Also we have

$$g\left(h(X, Y), N\right) = g(A_N X, Y), \qquad \text{..... (1.9)}$$

for any $X, Y \in \Gamma(TM), N \in \Gamma\left(TM^{\perp}\right)$.

2. Basic concepts.

For $X, Y \in \Gamma(TM)$, we denote

$$u(X, Y) = \nabla_X FPY - A_{FQY} X, \qquad \text{..... (2.1)}$$

and state

LEMMA 2.1 *Let M be a semi-invariant submanifold of the Kenmotsu manifold* $\overline{M}$. *Then we have*

$$P\left(u(X, Y)\right) = FP\nabla_X Y - \eta(Y) FPX, \qquad \text{..... (2.2)}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$Q(u(X,Y)) = Qt(h(X,Y)) \qquad \text{..... (2.3)}$$

$$\eta(u(X,Y)) = -g(X,FPY)\xi \qquad \text{..... (2.4)}$$

$$h(X,FPY) + \nabla_X^{\perp} FQY = FQ\nabla_X Y - \eta(Y)FQX + fh(X,Y) \qquad \text{..... (2.5)}$$

for all $X,Y \in \Gamma(TM)$.

Proof. By using the decompositions (1.5), (1.6) and the equations of Gauss and Weingarten in (1.3) we obtain

$$\nabla_X FPY + h(X,FPY) + \nabla_X^{\perp} FQY - A_{FQY}X = FP\nabla_X Y + FQ\nabla_X Y + th(X,Y) + fh(X,Y)$$
$$-g(X,FPY)\xi - \eta(Y)FPX - \eta(Y)FQX \qquad \text{..... (2.6)}$$

for all $X,Y \in \Gamma(TM)$.

Then (2.2) - (2.5) follows by taking the components on each of the vector bundle $D, D^{\perp}, \{\xi\}$ and respectively $TM^{\perp}$.

LEMMA 2.2. *Let M be a semi-invariant submanifold of the Kenmotsu manifold* $\overline{M}$, *then we have*

$$FP(A_N X) + P(\nabla_X tN) = P(A_{fN} X), \qquad \text{..... (2.7)}$$

$$Q\left(\left(t\nabla_X^{\perp} N\right) + A_{fN} X - \nabla_X tN\right) = O, \qquad \text{..... (2.8)}$$

$$\eta\left(A_{fN} X - \nabla_X tN\right) = g(X,tN), \qquad \text{..... (2.9)}$$

$$h(X,tN) + FQ(A_N X) + \nabla_X^{\perp} fN = f\nabla_X^{\perp} N, \qquad \text{..... (2.10)}$$

for any $X \in \Gamma(TM)$ *and* $N \in \Gamma(TM^{\perp})$.

Proof. By using the decompositions (1.5), (1.6) and the equations of Gauss and Weingarten in (1.3) we obtain

$$P(\nabla_X tN) + Q(\nabla_X tN) + \eta(\nabla_X tN)\xi + h(X,tN) - PA_{fN}X - QA_{fN}X - \eta(A_{fN}X)\xi$$
$$+\nabla_X^{\perp} fN + FP(A_N X) + FQ(A_N X) - Qt\nabla_X^{\perp} N = f\nabla_N^{\perp} - g(X,tN)\xi \qquad \text{..... (2.11)}$$

then (2.7) - (2.10) follows by taking the components on each of the vector bundle $D, D^{\perp}, \{\xi\}$ and respectively $TM^{\perp}$.

LEMMA 2.3 *Let M be an semi-invariant submanifold of the Kenmotsu manifold* $\overline{M}$, *then we have*

$$\nabla_X \xi = PX, h(X,\xi) = 0, \text{ for any } X \in \Gamma(D); \qquad \text{..... (2.12)}$$

$$\nabla_Y \xi = QX, h(Y,\xi) = 0, \text{ for any } Y \in \Gamma(D^{\perp}); \qquad \text{..... (2.13)}$$

$$\nabla_\xi \xi = 0, \ h(\xi,\xi) = 0. \qquad \text{..... (2.14)}$$

Proof. By using (1.5), and Gauss formula in (1.4) we obtain

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\overline{\nabla}_X \xi = X - \eta(X)\xi$$
$$= PX + QX + \eta(X)\xi - \eta(X)\xi$$

or, $\nabla_X \xi + h(X,\xi) = PX + QX$ ..... (2.15)

Now (2.12) - (2.14) follows from (2.15).

LEMMA 2.4. *Let M be a semi-invariant submanifolds of the Kenmotsu manifold* $\overline{M}$, *then we have*

$$A_{FX}Y = A_{FY}X, \text{ for all } X, Y \in \Gamma(D^{\perp}) \quad \text{..... (2.16)}$$

Proof. By means of (1.2), (1.3), (1.7), (1.9) we have

$$g(A_{FX}Y, Z) = g(h(Y,Z), FX) = g(\overline{\nabla}_Z Y, FX)$$
$$= -g(F\overline{\nabla}_Z Y, X) = -g(\overline{\nabla}_Z FY, X)$$
$$= g(FY, \overline{\nabla}_Z X) = g(FY, h(X,Z))$$
$$= g(A_{FY}X, Z)$$

for all $X, Y \in \Gamma(D^{\perp})$ and $Z \in \Gamma(TM)$, which implies (2.16).

LEMMA 2.5. *Let M be a semi-invariant submanifold of the Kenmotsu manifold* $\overline{M}$. *Then we have*

$$\nabla_\xi U \in \Gamma(D), \text{ for any } U \in \Gamma(D) \quad \text{..... (2.17)}$$

$$\nabla_\xi V \in \Gamma(D^{\perp}), \text{ for any } V \in \Gamma(D^{\perp}) \quad \text{..... (2.18)}$$

Proof. We take $X = \xi$ and $Y = U \in \Gamma(D)$ in (2.3) and obtain

$$Q\nabla_\xi FU = 0 \quad \text{..... (2.19)}$$

on the other hand (2.14) implies

$$g(\nabla_\xi U, \xi) = 0 \quad \text{..... (2.20)}$$

Hence (2.17) follows from (2.19) and (2.20) since F is an isomorphism of D.

Now we take $X = \xi$ and $V = FN$, where $N \in \Gamma(FD^{\perp})$ in (2.7). Taking account that $tN = FN$, $fN = 0$ and by using (2.12) we obtain

$$P\nabla_\xi V = -FP(A_N \xi) = 0. \quad \text{..... (2.21)}$$

Also from (2.9) we have

$$\eta(\nabla_\xi V) = 0, \text{ for all } V = FN \in \Gamma(D^{\perp})$$

where $N \in \Gamma(FD^{\perp})$. ..... (2.22)

Therefore (2.18) follows from (2.21) and (2.22).

COROLLARY 2.1. *Let M be a semi-invariant submanifold of the Kenmotsu manifold* $\overline{M}$. *Then we have*

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$[X;\xi] \in \Gamma(D), \text{ for any } X \in \Gamma(D) \qquad \text{..... (2.23)}$$

$$[Y,\xi] \in \Gamma(D^{\perp}), \text{ for any } Y \in \Gamma(D^{\perp}) \qquad \text{..... (2.24)}$$

3. Integrability of distributions on an semi-invariant submanifold in a Kenmotsu manifold

The purpose of this section is to study the integrability of all distributions involved in the definition of a semi-invariant submanifold. First we state.

THEOREM 3.1 *Let M be a semi-invariant submanifold in the Kenmotsu manifold M. Then the distribution D is never involutive.*

Proof. By using (2.12), we have

$$\begin{aligned} g([X,Y],\xi) &= g(\nabla_X Y - \nabla_Y X, \xi) \\ &= -g(Y,PX) + g(X,PY) \\ &= -g(Y,FX) + g(X,FY) \\ &= 2g(X,FY), \text{ for any } X,Y \in \Gamma(D) \end{aligned} \qquad \text{..... (3.1)}$$

since F is an isomorphism of D.Thus when $p \neq 0$, we take $X = FY$ with $Y \neq 0$ and from (3.1) follows the non-integrability of D.

From theorem 3.1 follows

COROLLARY 3.1. *The distribution $D \oplus D^{\perp}$ is never involutive.*

THEOREM 3.2. *The distribution $D \oplus \{\xi\}$ is involutive if and only if we have*

$$h(X,FY) = h(Y,FX), \text{ for all } X,Y \in \Gamma(D) \qquad \text{..... (3.2)}$$

Proof. From (2.5) we obtain

$$h(X,FY) = FQ\nabla_X Y + fh(X,Y), \qquad \text{..... (3.3)}$$

for any X, $Y \in \Gamma(D)$. Hence we have further

$$h(X,FY) - h(Y,FX) = FQ([X,Y]) \qquad \text{..... (3.4)}$$

for all X, $Y \in \Gamma(D)$ and the assertion of the theorem follows from (3.4) by using (2.17).

THEOREM 3.2 *The distribution $D^{\perp}$ is always involutive.*

Proof. First by using (2.13) we obtain

$$\begin{aligned} g([X,Y],\xi) &= g(\nabla_X Y - \nabla_Y X, \xi) \\ &= g(X,QY) - g(Y,QX) \\ &= g(X,Y) - g(Y,X) = 0 \end{aligned}$$

for any X, $Y \in \Gamma(D^{\perp})$.

Next from (2.1) follows $u(X, Y) = A_{FY} X$. Applying F to (2.2) we obtain

$$P\nabla_X Y = FPA_{FY} X, \text{ for any } X,Y \in \Gamma(D^{\perp}) \qquad \text{..... (3.6)}$$

Thus by using Lemma 2.4, from (3.6) we have

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$P([X,Y]) = FP(A_{FY}X - A_{FX}Y) = 0$$

for any $X, Y \in \Gamma(D^{\perp})$ ..... (3.7)

The assertion of the theorem follows from (3.5) and (3.7).

4. The geometry of leaves on a semi-invariant submanifold in a Kenmotsu manifold

The purpose of this section is to obtain results on the immersions of leaves of distributions in M or in $\overline{M}$. For the leaves of involutive distributions we shall obtain necessary and sufficient conditions in order that their immersions in M be totally geodesic.

THEOREM 4.1. *Let M be a semi-invariant submanifold of the Kenmotsu manifold $\overline{M}$. Then the distribution $D \oplus \{\xi\}$ is involutive and its leaves are totally geodesic in M if and only if*

$$h(X,Y) \in \Gamma(\nu), \text{ for any } X, Y \in \Gamma(D), \qquad \text{..... (4.1)}$$

and

$$\nabla_X Z \in \Gamma(D^{\perp}), \text{ for any } X \in \Gamma(D) \text{ and } Z \in \Gamma(D^{\perp}) \qquad \text{..... (4.2)}$$

Proof. By using (1.2), (1.3), (1.7), (1.8) we obtain

$$\begin{aligned} g(\nabla_X V, FY) &= g(\overline{\nabla}_X V, FY) \\ &= -g(F\overline{\nabla}_X V, Y) \\ &= -g(\overline{\nabla}_X FV, Y) \\ &= g(A_{FV}X, Y) \end{aligned} \qquad \text{..... (4.3)}$$

or, $$-g(\nabla_X FY, V) = g(h(X,Y), FV) \qquad \text{..... (4.4)}$$

for any $X, Y \in \Gamma(D \oplus \{\xi\})$ and $V \in \Gamma(D^{\perp})$.

Now suppose the distribution $(D \oplus \{\xi\})$ is involutive and its leaves are totally geodesic immersed in M. This happens if and only if we have

$$\nabla_X Y \in \Gamma(D \oplus \{\xi\}), \text{ for any } X, Y \in \Gamma(D \oplus \{\xi\}).$$

Thus (4.2) follows from (4.5) and (4.1) is a consequence of (4.4) by means of (4.5).

Conversely, suppose (4.1) and (4.2) be satisfied. From (2.14) and (2.17) follows

$\nabla_\xi Y \in \Gamma(D)$ for any $Y \in \Gamma(D \oplus \{\xi\}$ on the other hand from (2.12) we have ..... (4.6)

$$\nabla_X \xi = PX \in \Gamma(D) \text{ for any } X \in \Gamma(D). \qquad \text{..... (4.8)}$$

Finally by using (4.1) and (4.2) in (4.4) we obtain

$$g(\nabla_X FY, V) = 0, \text{ for any } X, Y \in \Gamma(D) \text{ and } V \in \Gamma(D^{\perp})$$

From (4.6), (4.7), (4.8) follows (4.5) and the proof is complete.

THEOREM 4.2. *Let M be a semi-invariant submanifold of the Kenmotsu manifold $\overline{M}$. Then any leaf of the distribution $D^{\perp}$ is totally geodesic in M if and only if*

$$h(X, FPZ) \in \Gamma(\nu), \qquad \text{..... (4.9)}$$

for any $X \in \Gamma(D^{\perp})$ and $Z \in \Gamma(D \oplus \{\xi\})$.

Proof. By using (1.2), (1.3), (1.7) and (1.9) we obtain

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$g(\nabla_X Y, Z) = g(\overline{\nabla}_X Y, Z) = -g(Y, \overline{\nabla}_X Z)$$
$$= -g(F\overline{\nabla}_X Z, FY) = -g(\overline{\nabla}_X FZ, FY)$$
$$= -g(\overline{\nabla}_X FPZ, FY) = g(\nabla_X FPZ + h(X, FPY), FY)$$
$$= g(h(X, FPZ), FY) \qquad \text{..... (4.10)}$$

for any $X, Y \in \Gamma(D^{\perp})$ and $Z \in \Gamma(D \oplus \{\xi\})$.

Thus $\nabla_X Y \in \Gamma(D^{\perp})$ if and only if (4.9) is satisfied and the proof is complete.

THEOREM 4.3. *Let M be a semi-invariant submanifold of the Kenmotsu manifold $\overline{M}$. Then the following assertion are equivalent to each other*

(i) *The distribution $D^{\perp} \oplus \{\xi\}$ is involutive and its leaves are totally geodesics in M.*

(ii) *For any $X, Y \in \Gamma(D^{\perp} \oplus \{\xi\})$, we have*

$$A_{CY} X \in \Gamma(D^{\perp} \oplus \{\xi\}) \qquad \text{..... (4.11)}$$

(iii) *For any $X \in \Gamma(D^{\perp} \oplus \{\xi\})$, $Z \in \Gamma(D)$ and $V \in \Gamma(D^{\perp})$, we have*

$$h(X, Z) \in \Gamma(\nu), \qquad \text{..... (4.12)}$$

and

$$\nabla_X V \in \Gamma(D^{\perp} \oplus \{\xi\}) \qquad \text{..... (4.13)}$$

Proof. We have the asertion (i) if and only if

$$g(\nabla_X Y, Z) = 0 \qquad \text{..... (4.14)}$$

for any $X, Y \in \Gamma(D^{\perp} \oplus \{\xi\})$ and $Z \in \Gamma(D)$. By using (1.2), (1.3), (1.5) and the equations of Gauss and Weingarten we obtain

$$g(\nabla_X Y, Z) = g(\overline{\nabla}_X Y, Z) = g(F\overline{\nabla}_X Y, FZ)$$
$$= g(\overline{\nabla}_X FY, FZ) = -g(A_{CY} X, FZ) \qquad \text{..... (4.15)}$$

for any $X, Y \in \Gamma(D^{\perp} \oplus \{\xi\})$ and $Z \in \Gamma(D)$. Thus equivalence of (i) and (ii) is proven by (4.15) taking account that D is invariant by F.

In a similar way we obtain

$$g(\nabla_X Y, Z) = -g(Y, \overline{\nabla}_x Z) =$$
$$= -g(FY, F\overline{\nabla}_x Z)$$
$$= -g(FY, \overline{\nabla}_X FZ)$$
$$= -g(FQY, \nabla_X FZ + h(X, FZ))$$

for any $X, Y \in \Gamma(D^{\perp} \oplus \{\xi\})$ and $Z \in \Gamma(D)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Suppose (4.14) be satisfied. If we take $Y = V \in \Gamma\left(D^{\perp}\right)$ in (4.14) we obtain (4.13). Then (4.12) follows from (4.16), since F is an automorphism of D.

Finally, assuming that the (4.12) and (4.13) are satisfied, then from (4.16) we obtain (4.14), showing that (i) and (iii) are equivalent.

References

Bejancu, A. and Papaghiuc, N. (1984) : Almost semi-invariant submanifolds of a Sasakian manifold; *Bull. Math. dela Soc. Sci. Math. dela* R. S. Roumanie 28(76), 1.

Kobayashi, M (1986) : Semi-invariant submanifolds of a certain class of almost contact manifolds; *Tensor*, N. S. 43.

Papaghiuc, N. (1984) : Some result on almost semi-invariant submanifolds in Sasakian manifolds. *Bull. Math. dela Soc. Sci. Math. dela R.S. Roumanie* 28(76) 3.

DEPARTMENT OF MATHEMATICS,
BANARAS HINDU UNIVERSITY
VARANASI 221 005, INDIA.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc. **86**, 413-418 (1994)

ON M-MANIFOLDS

SHANTESHWAR PRASAD AND RAM HIT OJHA

(Received 31 March 1992; revision received 30 September 1993)

Abstract : Mishra (1989) introduced some new manifolds like M-manifold, nearly M-manifold and generalised quasi Sasakian manifold. The purpose of this paper is to study some properties of M-manifolds.

1. Introduction. An odd dimensional differentiable manifold V_{2m+1} in which there are defined a tensor field F of type (1, 1) a vector field T, a 1-form A and a metric tensor g satisfying for arbitrary vector fields X, Y, Z...

(a) $\overline{\overline{X}} = -X + A(X)T$ (b) $\overline{T} = 0$

(c) $g(\overline{X}, \overline{Y}) = g(X, Y) - A(X)A(Y)$

(d) $'F(X,Y) = g(\overline{X}, Y) = -'F(Y,X) = 'F(\overline{X}, \overline{Y})$ ..... (1.1)

where

(a) $\overline{X} \overset{def}{=} F(X)$ (b) $A(X) = g(X,T)$ ..... (1.2)

is called an almost Grayan manifold. An almost Grayan manifold is said to be quasi Sasakian manifold if

$$(D_X{}'F)(Y,Z) + (D_Y{}'F)(Z,X) + (D_Z{}'F)(X,Y) = 0 \quad \text{..... (1.3)}$$

An almost Grayan manifold satisfying :

$$(D_X A)(\overline{Y}) = (D_{\overline{X}} A)(Y) = -(D_Y A)(\overline{X}) \quad \text{..... (1.4)}$$

(a) is called M-manifold if

$$(D_X F)Y = A(Y) D_{\overline{X}} T + (D_Y A)(\overline{X}) T, D_T F = 0 \quad \text{..... (1.5)}$$

(b) is called nearly M-manifold if

$$(D_X F)Y + (D_Y F)X = A(Y)(D_{\overline{X}} T) + A(X)(D_{\overline{Y}} T) \quad \text{..... (1.6)}$$

where D is a Reimannian connection.

Nijenhuis tensor with respect to F is a Vector valued bilinear function N given by

$$N(X,Y) \overset{def}{=} (D_{\overline{X}} F)Y - (D_{\overline{Y}} F)X + \overline{(D_Y F)X} - \overline{(D_X F)Y} \quad \text{..... (1.7)}$$

$$'N(X,Y,Z) = (D_{\overline{X}}{}'F)(Y,Z) - (D_{\overline{Y}}{}'F)(X,Z) + (D_X{}'F)(Y,\overline{Z}) - (D_Y{}'F)(X,\overline{Z}) \quad \text{..... (1.8)}$$

where $'N(X,Y,Z) \overset{def}{=} g(N(X,Y),Z)$

THEOREM 1.1 : *We have on M-manifold*

(a) $'N(X,Y,\overline{Z}) = 0$ (b) $'N(\overline{X}, \overline{\overline{Y}}, Z) - 'N(\overline{\overline{X}}, \overline{Y}, Z) = 0$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(c) $'N(\overline{X},\overline{Y},Z)+'N(\overline{\overline{X}},\overline{Y},Z)+'N(\overline{X},\overline{\overline{Y}},Z)=2A(Z)(D_{\overline{X}}A)(\overline{Y})-(D_{\overline{X}}A)(Y)$ ..... (1.9)

Proof : Form (1.8) and (1.5) we have

$$'N(X,Y,Z)=(2(D_XA)(Y)+A(X)(D_YA)(T)-A(Y)(D_XA)(T))A(Z) \quad \text{..... (1.10)}$$

Barring Z in (1.10) we get (1.9) (a). Barring X and Y in (1.10), we get

$$'N(\overline{X},\overline{Y},Z)=2A(Z)(D_XA)(Y) \quad \text{..... (1.11)}$$

Again barring X in (1.11) and using (1.4)

we get $'N(\overline{\overline{X}},\overline{Y},Z)=-2A(Z)(D_{\overline{X}}A)(Y)$

Similarly we get

$$'N(\overline{X},\overline{\overline{Y}},Z)=-2A(Z)(D_XA)(\overline{Y})$$

Using (1.4) we get (1.9) b.

By same computation we get (1.9) (c).

2. Affine Connections. Let D be a Riemannian connection on an M-manifold V_{2m+1} and B be another connection. Let us put

$$B_XY=D_XY+H(X,Y) \quad \text{..... (2.1)}$$

where $H(X,Y)$ is a bilinear function

Torson tensor S of B is given by

$$\begin{aligned} S(X,Y)&=B_XY-B_YX-[X,Y] \\ &=B_XY-B_YX-D_XY+D_YX \\ &=H(X,Y)-H(Y,X) \end{aligned} \quad \text{..... (2.2)}$$

If we assume that $H(X, Y)$ is skew symmetric

then

$$H(Y,X)=-H(X,Y)$$

$$S(X,Y)=2H(X,Y) \quad \text{..... (2.3)}$$

Consequently

$$'S(X,Y,Z)='H(X,Y,Z)-'H(Y,X,Z)=2'H(X,Y,Z) \quad \text{..... (2.4)}$$

where

$$'S(X,Y,Z)=g(S(X,Y),Z),\ 'H(X,Y,Z)=g(H(X,Y),Z) \quad \text{..... (2.5)}$$

THEOREM 2.1. *On M-manifold*

$$(B_XA)(\overline{Y})=(B_{\overline{X}}A)(Y)=-(B_YA)(\overline{X}) \quad \text{..... (2.6)}$$

if $A(H(X,\overline{Y}))=0$ *and* $H(X, Y)$ *is skew symmetric*

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. From (1.4) we have

$$(D_X A)(\overline{Y}) = -(D_Y A)(\overline{X})$$

$$\Rightarrow A(D_X \overline{Y}) = -A(D_Y \overline{X})$$

$$\Rightarrow A((B_X \overline{Y}) - H(X, \overline{Y})) = -A((B_Y \overline{X}) - H(Y, X))$$

$$\Rightarrow A(B_X \overline{Y}) + A(B_Y \overline{X}) = A(H(X, \overline{Y})) + A(H(Y, \overline{X}))$$

As $H(X, Y)$ is skew symmetric, we have

$$A(H(X, Y)) = -A(H(Y, X))$$

Hence

$$(B_X A)(\overline{Y}) + (B_Y A)(\overline{X}) = A(H(X, \overline{Y})) - A(H(X, \overline{Y})) = 0 \qquad \text{..... (2.7)}$$

Further as

$$(D_X A)(\overline{Y}) = (D_{\overline{X}} A)(Y)$$

$$\Rightarrow A(D_X \overline{Y}) = -\overline{X} A(Y) + A(D_{\overline{X}} Y)$$

$$\Rightarrow A(B_X \overline{Y} - H(X, \overline{Y})) = -\overline{X} A(Y) + A(B_{\overline{X}} Y - H(\overline{X}, Y))$$

$$\Rightarrow A(B_X \overline{Y}) - A(H(X, \overline{Y})) = -\overline{X} A(Y) + A(B_{\overline{X}} Y) - A(H(\overline{X}, Y))$$

$$(B_X A)(\overline{Y}) + (B_{\overline{X}} A)(Y) = A(H(X, \overline{Y}) - A(H(\overline{X}, Y))) \qquad \text{..... (2.8)}$$

Thus from (2.7) and (2.8) if $A(H(X, \overline{Y})) = 0$, we obtain (2.6).

THEOREM 2.2. *On M-manifold*

(a) $A(B_X \overline{\overline{Y}}) + A(B_{\overline{X}} \overline{Y}) = A(H(\overline{X}, \overline{Y})) - A(H(X, Y)) + A(Y) A(H(X, T))$

(b) $A(B_{\overline{X}} \overline{Y}) + A(H(X, Y)) = A(H(\overline{X}, \overline{Y})) + A(B_X Y)$ ..... (2.9)

Proof. From $(D_X A)(\overline{Y}) = (D_{\overline{X}} A)(Y)$, we have

$$A(D_X \overline{Y}) = \overline{X} A(Y) - A(D_{\overline{X}} Y)$$

Using (2.1) in the above eqation we get

$$A(B_X \overline{Y} - H(X, \overline{Y})) = \overline{X} A(Y) - A(B_{\overline{X}} Y - H(\overline{X}, Y))$$

$$\Rightarrow A(B_X \overline{Y}) + A(B_{\overline{X}} Y) = \overline{X} A(Y) + A(H(X, \overline{Y}) + A(H(\overline{X}, Y))$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Barring Y we get (2.9) (a) using (1.1) (a) in (2.9) (a) we get (2.9) (b).

THEOREM 2.3. *Let B satisfy*

$$(a)\ (B_X F) = 0\ (b)\ {}'H(X,Y,Z) + {}'H(X,Z,Y) = 0 \qquad \text{..... (2.10)}$$

Then B is metric

Proof. In consequence of (2.1) (a) we have

$$X({}'F(Y,Z) = {}'F(B_X Y,Z) + {}'F(Y,B_X Z)$$

$$= (D_X{}'F)(Y,Z) + F(D_X Y,Z) + {}'F(Y,D_X Z)$$

Using (2.10) in this equation, we get

$$= (D_X{}'F)(Y,Z) = H(X,Y,Z) - {}'H(X,Y,Z)$$

Using (2.10)(b) we get $(D_X{}'F)(Y,Z) = -2'H(X,Y,Z)$

Now $Xg(Y,Z) = (B_X g)(Y,Z) + g(B_X Y,Z) + g(Y,B_X Z)$

$$= g(D_X Y,Z) + g(Y,D_X Z)$$

Substituting (2.1) in this equation we get

$$(B_X g)(Y,Z) + {}'H(X,Y,Z) + {}'H(X,Z,Y) = 0$$

Then consequence of (2.1)(b) assume the form

$$B_X g = 0$$

COROLLARY 2.4. *Let D be a Riemannian connection in M-manifold V_{2m+1} and B be an affine connection satisfying.*

$$(a)\ (B_X{}'F)(\overline{Y},\overline{Z}) = 0\ \ (b)'H(X,\overline{Y},\overline{Z}) + {}'H(X,\overline{Z},\overline{Y}) = 0 \qquad \text{..... (2.11)}$$

Then

$$2'H(\overline{X},\overline{Y},\overline{Z}) = N(\overline{Z},\overline{Y},\overline{X}) \qquad \text{..... (2.12)}$$

Proof. By simple computation we get.

$$2'H(X,\overline{Y},\overline{Z}) = (D_X{}'F)(\overline{Y},\overline{Z})$$

Using (1.8) in this equation result follows.

3. **Properties** Let us consider

$$0 = (D_{\overline{X}} g)T,Z) = \overline{X}g(T,Z) - g(D_{\overline{X}}T,Z) - g(T,D_{\overline{X}}Z)$$

$$= \overline{X}A(Z) - g(D_{\overline{X}}T,Z) - A(D_{\overline{X}}Z)$$

$$= (D_{\overline{X}}A)(Z) - g(D_{\overline{X}}T,Z)$$

This implies

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\left(D_{\overline{X}}A\right)(Z) = g\left(D_{\overline{X}}T, Z\right)$$

Thus M-manifold and nearly M-manifold can also be written as

$$(D_X{}'F)(Y,Z) = A(Y)\left(D_{\overline{X}}A\right)(Z) + A(Z)(D_Y A)\left(\overline{X}\right), (D_T{}'F) = 0 \qquad \text{..... (3.1)}$$

$$(D_X{}'F)(Y,Z) + (D_Y{}'F)(X,Z) = A(Y)\left(D_{\overline{X}}A\right)(Z) + A(X)\left(D_{\overline{Y}}A\right)(Z). \qquad \text{..... (3.2)}$$

THEOREM 3.1. *A M-manifold is a quasi Sasakian Manifold.*

Proof : Using equation (3.1) and writing similarly two other equations and adding. We get

$$(D_X{}'F)(Y,Z) + (D_Y{}'F)(Z,X) + (D_Z{}'F)(X,Y) = A(X)\left(\left(D_{\overline{Z}}A\right)(Y) + (D_Z A)\left(\overline{Y}\right)\right)$$
$$+A(Y)\left(\left(D_{\overline{X}}A\right)g + (D_X A)\left(\overline{Z}\right)\right) + A(Z)\left((D_Y A)\left(\overline{X}\right) + \left(D_{\overline{Y}}A\right)(X)\right)$$

Using equation (1.4) we get the required rsult.

THEOREM 3.2. *A M-manifold is necessarily a nearly M-manifold.*

Proof : In M-manifold

$$(D_X{}'F)(Y,Z) = A(Y)\left(D_{\overline{X}}A\right)(Z) + A(Z)(D_Y A)\left(\overline{X}\right)$$

and $$(D_Y{}'F)(X,Z) = A(X)\left(D_{\overline{Y}}A\right)(Z) + A(Z)(D_X A)\left(\overline{Y}\right)$$

or,

$$(D_X{}'F)(Y,Z) + (D_Y{}'F)(X,Z) = A(Y)\left(D_{\overline{X}}A\right)(Z) + A(X)\left(D_{\overline{Y}}A\right)(Z) + A(Z)$$
$$\left(\left(D_Y A\left(\overline{X}\right) + (D_X A)\left(\overline{Y}\right)\right)\right)$$

But the two manfolds belong to a class of almost grayan manifold satisfying $(D_X A)\left(\overline{Y}\right) = \left(D_{\overline{X}}A\right)(Y) = -(D_Y A)\left(\overline{X}\right)$ which gives $A(Z)\left((D_Y A)\left(\overline{X}\right) + D_X A\right)\left(\overline{Y}\right)\right) = 0$ and hence a M-manifold is a nearly M-manifold.

THEOREM 3.3. *A normal nearly M-manifold is a M-manifold.*

Proof. For a nearly M-manifold

$$(D_X{}'F)(Y,Z) + (D_Y{}'F)(X,Z) = A(Y)\left(D_{\overline{X}}A\right)(Z) + \left(A(X)\left(D_{\overline{Y}}A\right)(Z)\right. \qquad \text{..... (3.3)}$$

We know (2) that an almost contact metric manifold is normal if

$$\left(D_{\overline{X}}{}'F\right)\left(\overline{Y}, Z\right) = (D_X{}'F)(Y,Z) + A(Y)\left(D_{\overline{X}}A\right)(Z)$$

Using (3.3) in above equation, we get (3.4)

$$\left(D_{\overline{Y}}{}'F\right)\left(\overline{X}, Z\right) = (D_X{}'F)(Y,Z) + A(Y)(D_X A)(Z)$$

Equation (3.4) and (3.1) prove the statement.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

References

Misra R. S. (1989) : On different almost Grayan Manifold, *Progress of Mathematics*, **23.**

Misra R. S. (1984) : *Structure on differentiable manifold and their application*, Chandrama Prakashan, Allahabad.

DEPARTMENT OF MATHEMATICS,
FACULTY OF SCIENCE
BANARAS HINDU UNIVERSITY
VARANASI-221005, (INDIA)

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc. **86**, 419-428 (1994)

ROBERTS' EXAMPLE AND THE AR-PROPERTY OF CONVEX SETS IN NON-LOCALLY CONVEX LINEAR METRIC SPACES

M. I. M. AL-ALI AND NGUYEN TO NHU

(Received 13 May 1993; revision received 15 June 1994)

1. Introduction. We say that a metric sapce X is an ANR if and only if for any metric space Y which contains X topologically as a closed subset there exist a neighbourhood U of X in Y and a map, called a retraction, $r : Y \to X$ such that $r(x) = x$ for every $x \in X$.

We say that X is an AR if in the above definition we can take $U = Y$. The AR-problem in linear metric spaces is stated as follows :

Problem 1.1. (The AR-problem) Is every convex set in a linear metric space an AR? see (Geoghegan 1979) (West 1990), Problems LS1, LS6.

The AR-problem is one of the most outstanding open problems in infinite dimensional topology. It has been being attacked strongly during the last two decades. The aim of this paper is to discuss some partial solutions of this problem.

Let us observe that for locally convex spaces Problem 1-1 was settled affirmatively by Dugundji in 1951. However it remains open for non-locally convex linear metric spaces and is one of the most resistant problems in AR-theory.

Problem 1.1 is extremely important because of the following two reasons :

1. By a theorem of Borsuk, (1967), every compact AR-space has fixed point property. Therefore a positive answer to Problem 1-1 would give a solution to the following conjecture of Schauder.

Schauder's conjecture 1.2. Every compact convex set in a linear metric space has the fixed point property?

Schauder's conjecture is very famous. It was posed by Schauder in the Scottish book in 1935 and despite great efforts by topologists for more than; half a century his conjecture is still unproved.

2. Torunczyk has shown that a positive answer to the AR-problem would also provide a solution to the following problem.

Problem 1.3. (i) Is every infinite dimensional compact convex set in a linear metric space homeomorphic to the Hilbert cube $Q = [0,1]^{\infty}$?

(ii) Is every complete separable infinite dimensional linear metric space homeomorphic to the Hilbert space?

Problem 1-3 was posed by Banach in early 1930's and is the most fundamental question in infinite dimensional topology : In fact infinite dimensional topology was born on the way of searching for a solution of this problem.

The AR-problem becomes very important because its solution will settle both Banach's and Schauder's problems which have been open for over six decades.

Notation and Conventions : In this paper all maps are assumed to be continuous. By a linear metric space we mean a topological linear space X which is metrizable. We write $\|x-y\| = \rho(x,y)$, where ρ is an invariant metric (Rolewicz 1972). We may assume that $\|.\|$ is monotonous, that is

$\|\lambda x\| \le \|x\|$ for every $x \in X$ and $\lambda \in \mathbb{R}$ with $|\lambda| \le 1$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

We call $\| \ \|$ an F-norm.

The zero element of X is denoted by θ. A locally convex space is a linear matric space which possesses a basis of neighbourhoods of θ consisting of convex sets.

Let E be a subset of a linear space X. By conv E we denote the convex hull of E and span E denotes the linear subspace of X spanned by E.

Let E be a subset of a metric space X and $x \in X$. We denote $\|x - E\| = \inf\{\|x - y\| : y \in E\}$.

We say that $\Phi : [0, \infty) \to [0, \infty)$ is an Orlicz function if and only if the following conditions hold :

(*i*) $\Phi(t) = 0$ if and only if $t = 0$;

(ii) Φ is an increasing function such that $\Phi(2t) \le M\Phi(t)$ for every $t \in [0, \infty)$, where M is a constant;

(iii) $\lim_{t \to \infty} \Phi(t)/t = 0$.

Let (X, μ) be a measure space and let Φ be an Orlicz function. We denote :

$$L_\Phi(\mu) = \left\{ f : X \to \mathbb{R} \int_X \Phi(|f|) d\mu < \infty \right\}.$$

The space $L_\phi(\mu)$ is equipped with the F-norm

$$\|f\| = \int_X \Phi(|f|)\, d\mu \text{ for } f \in L_\Phi(\mu).$$

Then $L_\Phi(\mu)$ is a complete linear metric space and is called an Orlicz space.

If $\Phi(t) = t^p, 0 < p < 1$, then we denote $L_\Phi(\mu) = L_p(\mu)$ and $L_0(\mu) = L_\Phi(\mu)$, where $\Phi(t) = t(1+t)^{-1}$ for $t \in [0, \infty)$.

If $X = [0, 1]$ and μ is the Lebesgue measure on $[0, 1]$ then we denote $L_\Phi(\mu) = L_0$. Observe that from definition we have :

$$\left\{ L_p = \left\{ f : [0,1] \to \mathbb{R},\ \int_0^1 |f(t)| dt < \infty \right\} \right. \text{ for } 0 < p < 1; \text{ and}$$

$$L_0 = \left\{ f : [0,1] \to \mathbb{R}, \int_0^1 \frac{|f(t)|}{1 + |f(t)|} dt < \infty \right\}.$$

For undefined notation, (Borsuk 1967), (Bessaga and Pelczynski 1975).

2. Characterizing ANR-spaces. The problem of determining a metric space is an ANR or not is, in general, very difficult. There are a lot of criteria for recognizing ANR-spaces (Borsuk 1967), (Bessaga and Pelczynski 1975), (Hu 1965). The following characterization of ANR-spaces is very useful in discovering ANR-spaces.

Let X be a metric space. For an open cover $\mathcal{U}$ of X let $\mathcal{N}(\mathcal{U})$ denote the nerve of $\mathcal{U}$ equipped with the Whitehead topology. Let $\{\mathcal{U}_n\}$ be a sequence of open covers of X. We say that $\{\mathcal{U}_n\}$ is a sequence if and only if

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\sup\{\text{diam } U : U \in \mathcal{U}_n\} \to 0 \text{ as } n \to \infty.$$

We denote

$$\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n \text{ and } \mathcal{K}(\mathcal{U}) = \bigcup_{n=1}^{\infty} \mathcal{N}(\mathcal{U}_n \cup \mathcal{U}_{n+1}).$$

For each $\sigma \in \mathcal{K}(\mathcal{U})$ we write

$$n(\sigma) = \sup\{n \in \mathbb{N} : \sigma \in \mathcal{N}(\mathcal{U}_n \cup \mathcal{U}_{n+1})\}.$$

We say that a map $f : \mathcal{U} \to X$ is a selection if and only $f(U) \in U$ for every $U \in \mathcal{U}$.

The characterization of ANR's can be simplified by the following theorem.

THEOREM 2.1. *A metric space X is an ANR if and only if there exist a zero sequence of open covers $\{\mathcal{U}_n\}$ of X together with a map $f : |\mathcal{K}(U)| \to X$ such that for any selection $g : \mathcal{U} \to X$ and for any sequence $\{\sigma_k\}$ with $n(\sigma_k) \to \infty$ we have diam $\left\{g\left(\sigma_k^0\right) \cup f(\sigma_k)\right\} \to 0$.*

Let X be a closed subset of a metric space Z. For every open set $U \subset X$ we put

(1) $\text{Ext } U = \{x \in Z : d(x, U) < d(x, X \setminus U)\}$.

Then we have, (Klee 1960).

(2) $\text{Ext } U \cap V = \text{Ext } U \cap \text{Ext} V$.

The proof of Theorem 2-1 uses the following fact established in (Hu 1965) Lemmas 4-3, 4-4, 4-5p. 127-128.

LEMMA. 2.2. *Let X be a closed subset of a metric space Z. Then for any sequence $\{\mathcal{U}_n\}$ of open covers of X there exist a sequence of neighbourhoods $\{W_n\}$ of X in Z and a locally finite open cover $\mathcal{V}$ of $W_1 \setminus X$ with the following properties :*

(i) *$d(x, X) < 1/n$ for every $x \in W_n$;*

(ii) *$\overline{W}_{n+1} \subset W_n$ for every $n \in \mathbb{N}$;*

(iii) *If $V \in \mathcal{V}$ and $V \cap \overline{W}_n \neq \phi$ then $V \subseteq W_{n-1}$ and there exist $\varphi(V) \in \mathcal{U}_n$ and a point $g(\varphi(V) \in \varphi(V)$ such that $V \subset Ext\varphi(V)$ and*

(3) *$d(x, g(\varphi(V)) < 5d(x, X) + diam\varphi(V)$ for every $x \in V$;*

Proof of Theorem 2.1. Theorem 2.1 is a consequence of Theorem 1.1 in (Nhu 1984), however as oberved by Luukkainen, inequality (3) was misformulated in (Nhu 1984). Therefore we give the proof here for convenient to the reader. The necessity of Condition 2.1 is quite simple : By Arens-Eells theorem, see (Bessaga and Pelczynski 1975), we may consider X as a closed subset of a normed space Z. Let W be a neighbourhood of X in Z and let $r : W \to X$ be a retraction. For each $n \in \mathbb{N}$ take an open cover $\mathcal{V}_n$ of X consisting of convex subsets in W such that :

(4) conv $V \subset W$ and diam r (conv V) $< 2^{-n}$ for every $V \in$ st $\mathcal{V}_n$;

(5) $\mathcal{V}_{n+1} < \mathcal{V}_n$ for every $n \in \mathbb{N}$.

Denote

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$\mathcal{U}_n = \{U = V \cap X : V \in \mathcal{V}_n\}$.

Let $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ and let $g : \mathcal{U} \to X$ be any selection. Using the convexity we extend g to a map $g : |\mathcal{N}(\mathcal{U})| \to W$. It is easy to see that the map $f = rg : |\mathcal{N}(\mathcal{U})| \to X$ satisfies the required condition.

Now we shall prove the sufficiency. Let $\{\mathcal{U}_n\}$ be a zero sequence of open convers of X satisfying the condition of Theorem 2.1. We shall show that X is an ANR.

Assume that X is a closed subset of a metric space Z. Using Lemma 2.2 we take a sequence of open neighbourhoods $\{W_n\}$ of X in Z and an open cover $\mathcal{V}$ of $W_1 \setminus X$ satisfying conditions 2.2(i)-(iii). We show that X is a retract of W_1.

For each $V \in \mathcal{V}$, put

$n(V) = \sup\{n : V \cap \overline{W}_n \neq \phi\}$.

By 3.2 (iii) there is a $\varphi(V) \in \mathcal{U}_{n(V)}$ and $g(\varphi(V)) \in \varphi(V)$ such that $V \subset Ext\ \varphi(V)$ and

(6) $d(x, g(\varphi(V)) < 5d(x, X) + \text{diam}\ \varphi(V)$ for every $x \in V$;

Observe that from (2) it follows that φ induces a simplicial map $\varphi' : \mathcal{N}(\mathcal{V}) \to \mathcal{N}(\mathcal{U})$.

Let $h : W_1 \setminus X \to \mathcal{N}(\mathcal{V})$ be the canonical map. We define a retraction $r : W_1 \to X$ by the formula :

$$r(x) = \begin{cases} x & \text{if } x \in X; \\ f(\varphi'(h(x)) & \text{if } x \in W_1 \setminus X \end{cases}$$

We shall show that r is continuous. For every $x \in W_1 \setminus X$, say $x \in \overline{W}_{n(x)} \setminus W_{n(x)+1}$, let $\sigma = < V_1, \ldots, V_n >$ be a simplex of $\mathcal{N}(\mathcal{V})$ containing $h(x)$. It is easy to see that

$\varphi(\sigma) \subset \mathcal{U}_{n(x)} \cup \mathcal{U}_{n(x)+1}$, where

$\varphi(\sigma) = < \varphi(V_1), \ldots, \varphi(V_{n+1}) > \in \mathcal{N}(\mathcal{U})$.

Thus we have $n(\varphi(\sigma)) \geq n(x)$. From (6) we get

$$\begin{aligned} d(x, r(x)) &= d(x, f\varphi' h(x)) \\ &\leq d(x, g(\varphi(V_1)) + d(g(\varphi(V_1)), f(\varphi(V_1)) + d(f(\varphi(V_1), f\varphi' h(x)) \\ &\leq 5d(x, X) + \text{diam}\varphi(V_1) + \text{diam}\{g(\sigma) \cup f(\sigma)\} + \text{diam}\, f(\sigma) \\ &\leq 5d(x, X) + \text{diam}\varphi(V_1) + 2\text{diam}\{g(\sigma) \cup f(\sigma)\}. \end{aligned}$$

Since $n(\varphi(\sigma)) \geq n(x) \to \infty$ as $x \to x_0 \in X$ we infer that r is continuous. The theorem is proved.

REMARK. Since $\{\mathcal{U}_n\}$ is a zero sequence the condition "for any selection g" in Theorem 2-1 is equivalent to the condition "for some selection g".

As a simple consequence of Theorem 2.1 we obtain Dugundji theorem.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

COROLLARY 2-3. (Dugundji 1951) *Every convex subset of a locally convex space is an AR.*

Proof. Let X be a convex set in a locally convex space. Since X is contractible, it suffices to show that X is an ANR.

Let $\{\mathcal{U}_n\}$ be a sequence of open convers of X consisting of covex subsets of X with the following properties :

(7) diam conv $U < 2^{-n}$ for every $U \in \text{st}\, \mathcal{U}_n$;

(8) $\mathcal{U}_{n+1} < \mathcal{U}_n$ for every $n \in \mathbb{N}$.

Let $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ and let $g : \mathcal{U} \to X$ be any selection. Using the convexity of X we extend g to a map $f : |\mathcal{N}(\mathcal{U})| \to X$. It is easy to see that the map satisfies the condition of Theorem 2.1; therefore X is an ANR and the corollary is proved.

3. Roberts example, needles points spaces and the finite dimensional approximation property. In 1940 Krein and Milman proved the following theorem :

THEOREM 3.1. (Krein and Milman 1940) *Any compact convex set in a locally convex space is the closure convex hull of its extreme points.*

The following question was open for a long time :

Question 3.2. Does Krein-Milman theorem holds true for non-locally convex linear metric spaces?

In 1976 Roberts constructed a striking example of a linear metric space which contains a compact convex set without any extreme points. Thus the Krein-Milman theorem does not hold true for non-locally convex linear metric spaces. Roberts proved the following theorem :

THEOREM 3.3. (Roberts 1976, 1977) *For every $p \in [0, 1)$ the space L_p contains a compact convex set with no extreme points.*

The idea of Roberts of constructing a compact convex set with no extreme points is to introduce the notion of needle point spaces.

DEFINITION 3.4. (Roberts 1976, 1977) *We say that a non-zero point a in a linear metric space X is a needle point iff for every $\varepsilon > 0$ there exists a finite set $A(a,\varepsilon) = \{a_1 \ldots a_m\}$ satisfying the following conditions :*

(i) $\|a_i\| < \varepsilon$ *for every* $i = 1, \ldots, m$;

(ii) *For every* $x \in \text{conv}\,\big(A(a,\varepsilon) \cup \{0\}\big)$ *there exists an* $\alpha \in [0,1]$ *such that* $\|x - \alpha a\| < \varepsilon$;

(iii) $a = m^{-1}(a_1 + \ldots + a_m)$.

A linear metric space X is a needle point space iff X is a complete separable space in which every non-zero point is a needle point.

Roberts proved following theorems

THEOREM 3.5. (Roberts 1976) *Every needle point space contains a compact convex set without any extreme points.*

THEOREM 3.6. (Roberts 1976) *For every $p \in [0, 1)$ the space L_p is a needle point space.*

For sometime it was hoped that Roberts' example would provide a counter-example to Schauder's conjecture. However this is not the case : In fact in (Nhu and Tri 1994) it is shown that *all compact convex sets* constructed by Roberts have the fixed point property. Let

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

us observe that in (Kalton, Peck and Roberts 1984) it was claimed that all compact convex sets constructed by Roberts' method have the fixed point property but no proof was given.

It was proved earlier in (Nhu and Tri 1994) that every needle point space contains a compact convex AR-set with no extreme points. However let us observe that the proof given in (Nhu and Tri 1994) has not yet reached *all Roberts' compact convex sets*. Some of Roberts compact convex sets were still standing away from the arguments given in (Nhu and Tri 1994).

After Roberts constructed his example needle point spaces become the most important area for finding a solution of Problem 1.1. It is hoped that needle point spaces (and in particular the spaces L_p, $0 \le p < 1$) will be a good place for constructing counter-examples to Problems 1.1. The following question arises naturally :

Problem 3.7. Is every convex set in a needle point space in AR?

We shall provide some partial answers to Problem 3.7. *The finite dimensinal approximation property* (the FDAP) is the key for the solution.

DEFINITION 3.8. (Nhu 1994) *Let X be a convex set in a linear metric space Y. We say that X has the finite dimensional approximation property (the FDAP) if and only if there exist an F-norm* $\| \ \|$ *on Y and a sequence of continuous maps* r_n *from X into finite dimensional subsets* X_n *of X such that for some summable sequence* $\{a_n\}$ *of positive numbers we have*

$$(FD) \lim_{n\to\infty} inf (a_n)^{-1}(1+\dim X_n)\|x - r_n(x)\| = 0 \text{ for every } x \in X.$$

THEOREM 3.9. (Nhu 1994) *Let X be a convex set in a linear metric space. If X has the FDAP then every convex subset* $E \subset X$ *is AR. In particular any convex set with the FDAP is an AR.*

Theorem 3.9 *reduces Problem 3-7 to*

Problem 3.10. Has every needle point spaces the FDAP?

As an application of Theorem 3-9 we obtain the following result which provides a particular answer to Problem 3-7.

THEOREM 3.11. (Nhu 1994) *Every needle point space X contains a dense linear subsapce* $E \subset X$ *with the following properties* :

(i) *E contains a compact convex set with no extreme points;*

(ii) *E has the FDAP, therefore every convex subset of E is an AR.*

Unfortunately, we are not able to prove that $E = X$. Even the answer to the following question has not yet been found :

Question 3.12. Is there a complete linear meteric space with properies (i) and (ii) of Theorem 3.11?

4. Examples of needle point spaces. The first needle point space was found by Roberts in (Roberts 1976). Later in (1977) Roberts showed that for any $p \in (0, 1]$ the space L_p is a needle point space. Other examples of needle point spaces can be found in (Kelton and Peck 1980), (Kelton, Peck and Roberts 1984). To prove that a given linear metric space is a needle point space is always a hard work.

In this section we establish the following theorem.

THEOREM 4.1. *Let* (X, μ) *be a normalized measure space and let* Φ *be an Orlicz function. Then* $L_\Phi(\mu)$ *contains a needle point space if and only if* μ *is not purely atomic.*

Our proof uses the main idea of Roberts in (1976). However we also add some new ideas. For instance Claims 4.3, 4.4. Only 4.3 and 4.4 are new. Moreover let us observe that the

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

proof of Roberts in (1976] does not describe a clear picture of a compact convex set with no extreme points even in the spaces L_p, $0 \le p < 1$, while our proof gives a concrete construction of such a compact convex set. Our result of this section extends the earlier result in (Nhu, Nhy and Tran 1994).

We begin with the proof of the necessity of Theorem 4.1.

THEOREM 5.2. *Let (X, μ) be a purely atomic normalized measure space. Then every compact convex set in $L_\Phi(\mu)$ is the closed convex hull of its extreme points.*

Proof. Since μ is normalized, the set $\mathscr{A}$ of all atom of X is finite or countable, (Halmos 1950). Therefore we have

$$\mathscr{A} = \{A_1, \ldots, A_k\}; k \le \infty;$$

$$\alpha_n = \mu(A_n) > 0 \text{ for } n = 1, \ldots k.$$

Since $X = \bigcup_{n=1}^{k} A_n$ we get

$$L_\Phi(\mu) = \left\{ x = (x_n) : \sum_{n=1}^{k} \alpha_n \Phi(|x_n|) < \infty \right\}.$$

Observe that the formula :

$$f_n(x) = x_n \text{ for every } x = (x_n) \in L_\Phi(\mu)$$

defines a sequence of continuous linear functions on $L_\Phi(\mu)$ which separates points of $L_\Phi(\mu)$. Therefore by Krein-Milman theorem, see (Rudian 1973), we get the assertion. The theorem is proved.

For the proof of the sufficiency of Theorem 4.1 we need several claims. In Claims 4.3-4.6 stated below we always assume that (X, μ) is an atomless normalized measure spaces.

Claim 4.3. For any $K \in \mathbb{N}$ there is a sequence $\{A_n\}$ of measurable sets in X satisfying the following conditions :

(i) $\mu(A_n) = K^{-1}$ for every $n \in \mathbb{N}$;

(ii) $\mu(A_n \cap A_m) = K^{-2}$ for every $n \ne m$.

Proof. Since X does not contain any atom, by (Halmos 1950) there exists a decomposition $X = \bigcup_{i=1}^{K} X_i$ such that $X_i \cap X_j = \phi$ for every $i \ne j$ and $\mu(X_i) = K^{-1}$ for every $i = 1, \ldots, K$.

Let $A_1 = X_1$ and decompose each set X_i into pairwise disjoint subsets $\{X_{ijk} : k = 1, \ldots, K\}$ of measure K^{-2}. Put $A_2 = \bigcup_{i=1}^{K} X_{i1}$. Obviously $\mu(A_2) = K^{-1}$ and $\mu(A_2 \cap A_1) = K^{-2}$.

Continuing, we decompose each set X_{ij} into K pairwise dsjoijnt subsets $\{X_{ijk} : k = 1, \ldots K\}$ of measure K^{-3} and put $A_3 = \bigcup_{i,j=1}^{K} X_{ij1}$. Then $\mu(A_3) = K^{-1}$ and $\mu(A_3 \cap A_i) = K^{-2}$ for $i = 1, 2$.

Finally we get the desired $\{A_n\}$ by induction.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Claim 4.4. Let A_1, A_2 be measureable sets of X such that $\mu(A_i) = \alpha$, $i = 1, 2$ and $\mu(A_1 \cap A_2) = \alpha^2$. Put $f_i = \alpha^{-1}\chi_{A_i}$, $i = 1,2$. Then

$$\int_X (f_1 - 1)(f_2 - 1)\, d\mu = 0.$$

Proof. Let $f = (f_1 - 1)(f_2 - 1)$ and write

$$\int_X f d\mu = \int_{A_1 \cap A_2} f d\mu + \int_{A_1 \setminus A_1 \setminus A_2} f d\mu + \int_{A_2 \setminus A_1 \cap A_2} f d\mu + \int_{X \setminus A_1 \cup A_2} f d\mu$$

An easy computation shows that

$$\int_{A_1 \cap A_2} f d\mu = \int_{X \setminus A_1 \cup A_2} f d\mu = (1-\alpha)^2;$$

$$\int_{A_1 \setminus A_1 \cap A_2} f d\mu = \int_{A_2 \setminus A_1 \cap A_2} f d\mu = -(1-\alpha)^2.$$

This claim 4.4.

Claim 4.5. For every $\delta > 0$ and $c_1 > 0$ there exists a sequence of simple functions $\{f_n\} \subset L_\Phi(\mu)$ and a positive number $c_2 < c_1$ such that for any finitely non-zero sequence $\{\alpha_n\}$ with $\alpha_n \geq 0$ and $\sum \alpha_n \leq 1$ we have

(i) $$\left\| \sum_{\alpha_n \geq c_1} \alpha_n f_n \right\| < \delta;$$

(ii) $$\left\| \sum_{\alpha_n \leq c_2} \alpha_n (f_n - 1) \right\| < \delta.$$

Proof. By Claim 4.3 for every $K \in \mathbb{N}$ there is a sequence $\{A_n\}$ of measurable sets of X such that $\mu(A_n) = K^{-1}$ for every $n \in \mathbb{M}$ and $\mu(A_n \cap A_m) = K^{-2}$ for $n \neq m$. Let us put $f_n = K_{\chi_{A_n}}$ for $n \in \mathbb{N}$. Then we have

$$\|f_n\| = \Phi(K)K^{-1} \text{ for every } n \in \mathbb{N}.$$

Therefore if we choose K sfficiently large then (i) holds.

To show that there exists $c_2 < c_1$ satisfying conditon (ii) first we use Jensen's ineqality to obtain

$$\left\| \sum_{n=1}^{k} \alpha_n (f_n - 1) \right\| \leq \phi\left(\int_X \left[\sum_{n=1}^{k} \alpha_n^2 (f_n - 1)^2 \right]^{1/2} \right).$$

Using Claim 4.4 we get

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\int_X \left(\sum_{n=1}^{k} \alpha_n (f_n - 1) \right)^2 d\mu = \int_X \sum_{n=1}^{k} \alpha_n^2 (f_n - 1)^2 \, d\mu.$$

Hence

$$\left\| \sum_{n=1}^{k} \alpha_n (f_n - 1) \right\| \le \Phi\left(\left[Kk^{-1} \right]^{1/2} \right).$$

Therefore taking $c_2 = (k_0)^{-1}$ for sfficiently large $k_0 \in \mathbb{N}$ we obtain condition (ii) and the claim is proved.

THEOREM 4.6. *$L_\Phi(\mu)$ is a needle point space.*

Proof. It is easy to see that the needle point property of the function $f_0(t) = 1$ for every $t \in [0,1]$ implies the needle point property of all other points of $L_\Phi(\mu)$. Therefore we have only to prove that f_0 is a needle point of $L_\Phi(\mu)$.

Given $\varepsilon > 0$ we take sequence of positive numbers $\{\delta_i\}$ such that $\sum_{i=1}^{\infty} \delta_i < \varepsilon / 3$. By claim 4.5 there exists a decreasing seqence of positive numbers $\{c_i\}$ with $c_1 = 1$ such that for each $i \in \mathbb{N}$ there is a sequence of simple fnctions $\left\{ f_n^i \right\}$ such that for any finitely non-zero sequence $\{\alpha_n\}$ with $\alpha_n \ge 0$ and $\sum \alpha_n \le 1$ we have :

(9) $$\left\| \sum_{\alpha_n \ge c_i} \alpha_n f_n^i \right\| < \delta_i;$$

(10) $$\left\| \sum_{\alpha_n \le c_{i+1}} \alpha_n \left(f_n^i - 1 \right) \right\| < \delta_i.$$

Choose $K \in \mathbb{N}$ so that

(11) $$sup\left\{ \left\| K^{-1} f \right\| : \|f\| \le 1 \right\} < \varepsilon / 3.$$

Denote

$$a_n = K^{-1} \sum_{i=1}^{K} f_n^i \quad \text{for } n = 1, 2, \ldots$$

and take $m \in \mathbb{N}$ so that $m > (c_{K+1})^{-1}$.Then $\{a_1, \ldots, a_m\}$ satisfies conditions 3.4 (i), (ii) and

(12) $$\left\| f_0 - \frac{1}{m} (a_1, \ldots, a_m) \right\| < \varepsilon.$$

It is easy to see that conditions 3.1 (iii) can be replaced by (12). Therefore Theorem 4.6 is proved.

Finally, let us observe that Theorem 4.1 follows from theorem 4.2 and Theorem 4.6.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

References

Bessaga, C. and Pelczynski, A. (1975) : *Selected topics in infinite dimensional topology,* Warszawa.

Borsuk, K. (1967) : *Theorey of retracts,* Warszawa.

Dobrowolski, T. and Torunczyk, H. (1979) : On linear metric spaces homeomorphic to ℓ_2 and compact onvex sets homeomorphic to *Q, Bull. Pol. Acad. Sci. Ser. Sci. Math.* **27**, 883.

Dobrowoski, T. and Torunczyk, H. (1981) : Separable complete ANR's admitting a group structure are Hilbert manifolds, *Top. Appl.* **12**, 229.

Dugundji, J. (1951) : An extension of Tietze's theorem, *Pacific J. Math.* **1**, 353.

Dugundji, J. and Granas A. (1982) : Fixed point theory, I, Warszawa.

Geoghegan, R. (1979) : Open problems in infinite dimensional topology, *Topology Proc.* **4**, 287.

Halmos, P. R. (1950) : *Measure theory,* Toronto-New York-London.

Hu, S. T. (1965) : *Theory of retracts,* Detroit.

Kalton, N. J. and **Peck, N. T.** (1980) : A re-examination of Roberts' example of a compact convex sets with no extreme points, *Math. Ann.* **253**, 89.

Kalton, N. J., Peck N. T. and **Roberts, J. W.** (1984) : An F-space sampler, *London Math.Soc. Lecture Note Series* **89**.

Klee, V. (1960) : Shrinkable neighbourhoods in Hausdorff linear spaces, *Math. Ann.* **141**, 281.

Klee, V. (1960) : Leray-Schader theory without local convexity, *Math. Ann.* **141**, 286.

Kuratowski, K. (1968) : *Topology,* II, New York.

Van Mill, J. (1989) : *Infinite dimensional topology, Prerequisites and introduction*, North-Holland, Amsterdam,

Nhu, Nguyen To (1984) : Investigating the ANR-property of metric spaces, *Fund. Math.* **124**, 243; *Correction; Fund. Math.* **141 (1992), 297.**

Nhu, Nguyen To (1987) : Remarks on measrable function spaces, *Acta Math. Vietnam* **12**, 85.

Nhu, Nguyen To (1994) : The AR-property in linear metric spaces, Top. Appl. (to appear).

Nhu Nguyen To, Nhuy Nguyen and **Van An, Tran,** (1994) : *Regular retraction onto finite dimensional convex sets and the AR-property for Roberts spaces, Preprint.*

Nhu, Nguyen To and Tri, Le Hoang (1994) : Every needle point space contains a compact convex AR-set with no extreme points, *Proc. Amer. Math. Soc.* **120**, 1261.

Nhu, Nguyen To and **Tri, Le Hong** : (1994) : No Roberts space is a counter-example to Schauder's conjectue, *Topology,* 33, 371.

Roberts, J. W. (1977) : A compact convex set with no extreme points, *Studia Math.* **60**, 255.

Roberts, J. W, (1976) : *Pathological compact convex sets in the spaces L_p $0 \le p < 1$,* The Altgeld Book, University of Illinois,

Rolewicz, S. (1972) : *Metric linear spaces,* Warszawa.

Rudin, W. (1973) : *Fnctional analysis,* Mc. Gra-Hill.

Torunczyk, H. (1980) : On CE-images of Hilbert cube and characterization of Q-manifolds, *Fund. Math.* **106**, 31.

Torunczyk, H. (1980) : Characterization of infinite dimensional manifolds, *Proceedings of the international conference on geometric topology,* Warszawa, 431.

Torunczyk, H. (1981) : Characterizing Hilbert space topology, *Fund. Math.* **111**, 247.

West, J.E. (1990) : *Open problems in infinite dimensional topology, in Open problems in topology,* J. van Mill and G.M. Reed (Editors), Elsevier Science Publishers B.V. (North-Holland), 523.

DEPARTMENT OF MATHEMATICS,
MU'TAH UNIVERSITY P.O. BOX-7,
MU'TAH-AL-KARAK JORDAN.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc. **86**, 429-436 (1994)

FUZZY LINEAR OPERATORS AND FUZZY NORMED LINEAR SPACES

SHIH-CHUAN CHENG AND JOHN N. MORDESON

(Received 13 May 1993; revision received 5 November 1993)

Abstract : We introduce the concepts of a fuzzy linear operator and fuzzy normed linear space. We determine the algebraic structure of the set of all fuzzy linear operators of a linear space into another. We define and characterize continuity of a fuzzy linear operator in terms of a fuzzy norm.

Introduction. The notion of a fuzzy linear space was introduced by (Katsaras and Liu 1977). (Katsaras 1984) and (Wu Congxon and Fang Ginxuan 1984) introduced the idea of fuzzy norms. Other papers have appeared to examine various fuzzy topological vector spaces, fuzzy normed linear spaces, and fuzzy inner product spaces (Biswas 1991), (El-Abyad & El-Hamouly 1991), (Katsaras 1981), (Khan & Ahsanullah 1993), (Kramosil & Michalek 1975), (Krishna & Sarma 1991), (Morsi 1988), (Wu & Fang 1984), and (Wu & Ma 1990). The purpose of this paper is to introduce the concept of a fuzzy linear operator. We determine the algebraic structure of the set of all fuzzy linear operators of a linear space into another, Theorems 1.10 and 1.11. We then introduce the concept of a fuzzy norm on a linear space. Our definition of a fuzzy norm allows the natural definition of a fuzzy metric via the fuzzy norm to yield the fuzzy metric of [(Kramosil & Michalek 1975) Definition 7, p. 341]. We define continuity for a fuzzy liner operator f in terms of a fuzzy norm and show that the continuity of f at one point is equivalent to the continuity of f everywhere which is in turn equivalent to the boundedness of f, Theorem 2.6. We recall that a fuzzy subset of a set U is a function from U into the closed interval [0, 1] (or a complete distributive lattice L such that no two nonzero elements of L intersect in 0).

1. Fuzzy linear operators. Let U and V be vector spaces over a field F. Consider the following conditions on a fuzzy subset f of the cartesian-cross product $U \times V$:

(1) $\forall x \in U, \exists y \in V$ such that $f(x,y) > 0$;

(2) $\forall y \in V, \exists x \in U$ such that $f(x,y) > 0$;

(3) $\forall x \in U, \forall y_1, y_2 \in V, f(x,y_1) > 0$ and $f(x,y_2) > 0$ implies $y_1 = y_2$;

(4) $\forall x_1, x_2 \in U, \forall y \in V, f(x_1,y) > 0$ and $f(x_2,y) > 0$ implies $x_1 = x_2$,

DEFINITION 1.1. *Let f be a fuzzy subset of $U \times V$. If condition (3) holds, then f is called a fuzzy function of U into V. If conditions (3) and (4) hold, then f is called a one-to-one fuzzy function of U into V.*

DEFINITION 1.2. *Let f be a fuzzy subset of $U \times V$. Define the* ***domain*** *of f, written $\mathscr{D}(f)$, to be $\{x \in U \mid \exists y \in V, f(x,y) > 0\}$. Define the* ***image*** *of f, written $\mathscr{I}(f)$, to be $\{y \in V \mid \exists x \in U, f(x,y) > 0\}$ If $\mathscr{I}(f) = V$, then f is said to map $\mathscr{D}(f)$* ***onto*** *V.*

Our choice of definition of fuzzy function coincides with the one appearing in (Klir & Folger 1988, p. 73). It has the advantage over many other definitions in that the

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

composition of two fuzzy functions under our definition is again a fuzzy function (Malik & Mordeson 1992). Thus the isomorphism theorems for fuzzy linear operators can be fuzzified. Also the space of all fuzzy linear operators of a linear space into another can thus be viewed as an algebra.

DEFINITION 1.3. *Let f be a fuzzy function of U into V. Then f is called fuzzy linear operator of U into V if and only if* $\forall x_1, x_2 \in, U, \forall y \in V, \forall c \in F$,

(i) $f(x_1 + x_2, y) \geq \sup\{\inf\{f(x_1, y_1), f(x_2, y_2)\} | y = y_1 + y_2\}$;

(ii) $f(cx_1, cy) \geq f(x_1, y)$.

We let $\mathcal{FL}(U, V)$ denote the set of all fuzzy linear operators of U into V.

Suppose that $f \in \mathcal{FL}(U,V)$ and $c \in F$, $c \neq 0$. Then $f(x,y) = f\left(c^{-1}(cx), c^{-1}(cy)\right) \geq f(cx, cy)$. Thus for $c \neq 0$, $f(cx, cy) = f(x, y)$.

The proof of the following result is routine.

PROPOSITION. 1.4. *Let* $f \in \mathcal{FL}(U,V)$. *Then* $\mathcal{D}(f)$ *and* $\mathcal{J}(f)$ *are subspaces of U and V, respectively.*

Unless otherwise specified, we assume that $\mathcal{D}(f) = U$ where $f \in \mathcal{FL}(U,V)$.

PROPOSITION 1.5. (Malik & Mordeson 1992) *Let* $f \in \mathcal{FL}(U,V)$. *Then*

(i) $f(0,0) > 0$;

(ii) $\forall x \in U, \forall y \in V, f(x,y) > 0$ *implies* $f(-x,-y) > 0$.

PROPOSITION 1.6. *Let* $f \in \mathcal{FL}(U,V)$. *Let* $f^* = \{x \in U | f(x,0) > 0\}$. *Then* f^* *is a subspace of U.*

Proof. That $(f^*, +)$ is a commutative group follows from [(Malik & Mordeson 1992) Proposition 2.4, p. 141]. Let $c \in F$ and let $x \in f^*$. Then $f(cx, 0) \geq f(x, 0) > 0$. Thus $cx \in f^*$.

PROPOSITION 1.7. (Malik & Mordeson 1992) *Let* $f \in \mathcal{FL}(U.V)$. *T then* $f^* = \{0\}$ *if and only if f is one-to-one.*

DEFINITION 1.8. *Define + on* $\mathcal{FL}(U, V)$ *by* $\forall f, g \in \mathcal{FL}(U,V), (f+g)(x,y) = \sup\{\inf\{\{f(x,y_1)\ \ g(x,y_2)\}\} | y = y_1 + y_2\}$. *Define* $\bullet$ *between F and* $\mathcal{FL}(U, V)$ *by* $\forall c \in F, c \neq 0$, and $\forall f \in \mathcal{FL}(U,V)$, $(cf)(x,y) = f\left(x, c^{-1}y\right)$ *and* $(0f)(x, y) = 0$ *if* $y \neq 0$ *and* $(0f)(x, y) = \sup\{f(x,z) | z \in V\}$ *if* $y = 0$.

We note that $\exists$ unique $z \in V$ such that $f(x, z) > 0$ and so $(0f)(x, 0) = f(x, z)$.

PROPOSITION 1.9. *Let* $f, g \in \mathcal{FL}(U, V)$ *and let* $c \in F$. *Then*

(i) $(f+g)(x,y) = \inf\{f(x,y_1), g(x,y_2)\}$ *if* $y = y_1 + y_2, f(x,y_1) > 0$, and $g(x,y_2) > 0$; otherwise $(f+g)(x,y) = 0$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(ii) $(cf)(x, cy) = f(x, y)$ if $c \neq 0$.

Proof. (i) Given $(x, y) \in U \times V$, $\exists$ unique $y_1, y_2 \in V$ such that $y = y_1 + y_2, f(x, y_1) > 0$, and $g(x, y_2) > 0$ when $(f + g)(x, y) > 0$.

(ii) Immediate.

For $t \in [0, 1]$, let $\mathscr{FL}_t(U,V) = \{f \in \mathscr{FL}(U,V) | f(U \times V) = \{0,t\}\}$. Let $\mathscr{FL}^*(U,V) = \bigcup_{t\in[0,1]} \mathscr{FL}_t(U,V)$.

THEOREM 1.10. (i) $(\mathscr{FL}(U, V), +)$ *is a commutative semigroup with identity.*

(ii) $(\mathscr{FL}_t(U,V),+)$ *is a commutative group;*

(iii) $(\mathscr{FL}^*(U,V),+)$ *is a completely regular semigroup (Petrich 1973, Definition 1.1, p.104),*

Proof. (i) Let $f, g \in \mathscr{FL}(U,V)$. Then $(f + g)(x_1 + x_2, y) = \inf\{f(x_1+x_2, y'), g(x_1+x_2, y'')\}$ where $y = y' + y''$, $f(x_1 + x_2, y') > 0$, and $g(x_1 + x_2, y'') > 0$. Now $\inf\{f(x_1+x_2, y'), g(x_1+x_2, y'')\} \geq \inf\{\sup\{\inf\{f(x_1, y_1'), f(x_2, y_2')\} | y' = y_1' + y_2'\}, \sup\{\inf\{g(x_1, y_1''), g(x_2, y_2'')\} | y'' = y_1'' + y_2''\}\} = \inf\{\inf\{f(x_1, y_1'), f(x_2, y_2')\}, \inf\{g(x_1, y_1''), g(x_2, y_2'')\}\} = \inf\{(f+g)(x_1, y_1), (f+g)(x_2, y_2)\} = \sup\{\inf\{(f+g)(x_1, y_1), (f+g)(x_2, y_2)\} | y = y_1 + y_2\}$ where $f(x_1, y_1') > 0, f(x_2, y_2') > 0, g(x_1, y_1'') > 0, g(x_2, y_2'') > 0, y_1 = y_1' + y_1''$, and $y_2 + y_2' + y_2''$. Let $c \in F$, $c \neq 0$. Then $(f + g)(cx, cy) = \inf\{f(cx, y_1), g(cx, y_2)\}$ *if* $cy = y_1 + y_2$, $f(cx, y_1) > 0$, and $g(cx, y_2) > 0$. Now $\inf\{f(cx, y_1), g(cx, y_2)\} = \inf\{f(x, c^{-1}y_1), g(x, c^{-1}y_2)\} = (f+g)(x, y)$ since $cy = y_1 + y_2$ if and only if $y = c^{-1}y_1 + c^{-1}y_2$. Now suppose that $c = 0$. Then $(f + g)(0x, 0y) = (f + g)(0, 0) = \inf\{f(0, 0), g(0, 0)\} \geq \inf\{f(x, y_1), g(x, y_2)\} = (f + g)(x, y)$ where $y = y_1 + y_2$. Thus $f + g \in \mathscr{FL}(U, V)$. The proofs that $+$ is associative and commutative are routine. Let $t \in (0, 1]$. Define the fuzzy subset θ_t of $U \times V$ by $\theta_t(x, y) = 0$ if $y \neq 0$ and $\theta_t(x, 0) = t$. Then $\theta_t(x_1 + x_2, y) = 0 = \inf\{\theta_t(x_1, y_1), \theta_t(x_2, y_2)\}$ if $y_1 + y_2 \neq 0$ since then either $y_1 \neq 0$ or $y_2 \neq 0$ where $y = y_1 + y_2$. If $y = y_1 + y_2 = 0$, then $\theta_t(x_1 + x_2, y) = t \geq \inf\{(x_1, y_1)(x_2, y_2)\} = \sup\{\inf\{\theta_t(x_1, y_1'), \theta_t(x_2, y_2')\} | y = y_1' + y_2'\}$. Let $c \in F$. Then $\theta_t(cx, cy) = 0 = \theta_t(x, y)$ if $cy \neq 0$. Suppose that $cy = 0$. Then $\theta_t(cx, cy) = t \geq \theta_t(x, y)$. Thus $\theta_t \in \mathscr{FL}(U,V)$. If $t = 1$, then θ_t is the identity of $\mathscr{FL}(U, V)$.

(ii) The fuzzy linear operator θ_t is the identity of $\mathscr{FL}_t(U, V)$. For $f \in \mathscr{FL}_t(U, V)$, $-f \in \mathscr{FL}_t(U, V)$ where $-f$ is the fuzzy subset of $U \times V$ defined by $(-f)(x, y) = f(x, -y)$. $-f$ is the inverse of f.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(iii) $\mathscr{FL}^{*}(U, V)$ is a union of disjoint commutative groups, [(Petrich 1973) Theorem 1.6, p. 105].

THEOREM 1.11. *Let $t \in (0, 1]$. Then $\mathscr{FL}(U, V)$, $\mathscr{FL}_t(U, V)$, and $\mathscr{F}^{*}\mathscr{L}(U, V)$ are closed under multiplication by elements of $F \bullet \forall f, g \in \mathscr{FL}(U,V)$ and $\forall, c, d \in F$,*

(i) $(cd)f = c(df)$;

(ii) $c(f+g) = cf + cg$;

(iii) $(c+d)f = cf + df$;

(iv) $If = f$.

Proof. Let $f \in \mathscr{FL}(U,V)$ and let $c \in F$. Suppose that $c \neq 0$. Then $(cf)(x_1 + x_2, y) = f(x_1 + x_2, c^{-1}y) \geq \sup\left\{\left\{\inf\left\{f\left(x_1, c^{-1}y_1\right), f\left(x_2, c^{-1}y_2\right)\right\}\right| y = y_1 + y_2\right\} = \sup\left\{\inf\left\{(cf)(x_1, y_1)(cf), (x_2, y_2)\right\}\right| y = y_1 + y_2\}$. Suppose that $c = 0$. Then $(0f)(x_1 + x_2, y) = 0$ if $y \neq 0$ and $= f(x_1 + x_2, z) > 0$ for some $z \in V$ if $y = 0$. Now $\sup\left\{\inf\left\{(0f)(x_1, y_1), (0f)(x_2, y_2)\right\}\right| y = y_1 + y_2\} = 0$ *if* $y_1 \neq 0$ or $y_2 \neq 0$ and $= \inf\{f(x_1, z_1), f(x_2, z_2)\} > 0$ for some $z_1, z_2 \in V$ if $y_1 = y_2 = 0$. *If* $y \neq 0$, then $y_1 \neq 0$ or $y_2 \neq 0$ and so $(0f)(x_1 + x_2, y) = 0 = \sup\left\{\inf\left\{(0f)(x_1, y_1), (0f)(x_2, y_2)\right\}\right| y = y_1 + y_2\}$. Suppose that $y_1 = y_2 = 0$. Suppose that $f(x_1 + x_2, z) > 0$, $f(x_1, z_1) > 0$, and $f(x_2, z_2) > 0$. Then $z = z_1 + z_2$ else $f(x_1 + x_2, z_1 + z_2) = 0 \not\geq \sup\left\{\inf\left\{f(x_1, z_1'), f(x_2, z_2')\right\}\right| z_1 + z_2 = z_1' + z_2'\}$. Thus $f(x_1 + x_2, z) \geq \inf\left\{f(x_1, z_1), f(x_2, z_2)\right\}$. Now let $d \in F$. Suppose that $c \neq 0$. Then $(cf)(dx, dy) = f\left(dx, dc^{-1}y\right) \geq f\left(x, c^{-1}y\right) = (cf)(x, y)$. Suppose that $c = 0$ and $d \neq 0$. Then $(0f)(dx, dy) = 0$ if $dy \neq 0$ and $= f(dx, z) > 0$ for some $z \in V$ if $dy = 0$. Thus $(0f)(x, y) = 0$ if $y \neq 0$ and $= f(x, z') > 0$ for some $z' \in V$ if $y = 0$. Now $dy = 0$ iff $y = 0$. If $f(dx, z) > 0$ and $f(x, z') > 0$, then $z = dz'$ since $f(dx, dz') = f(x, z') > 0$ and so $(0f)(dx, dy) \geq (0f)(x, y)$. Suppose that $c = 0$ and $d = 0$. Then $(0f)(0x, 0y) = f(0, z) > 0$ for some $z \in V$ and so $z = 0$. Thus $(0f)(0x, 0y) = f(0, 0) \geq f(x, y)$. Hence $cf \in \mathscr{FL}(U, V)$.

If $f \in \mathscr{FL}_t(U, V)$, then $cf(U \times V) = \{0, t\}$ and so $cf \in \mathscr{FL}_t(U, V)$.

(i) The argument is routine if $c \neq 0 \neq d$. If either $c = 0$ or $d = 0$, then $(cd)f(x, y) = 0$ if $y \neq 0$ and $= \sup\left\{f(x,z) \middle| z \in V\right\}$ if $y = 0$. Suppose that $c = 0 \neq d$. Then $\left(c(df)\right)(x, y) = 0$ if $y \neq 0$ and $= \sup\left\{(df)(x,z) \middle| z \in V\right\} = \sup\left\{f\left(x, d^{-1}z\right) \middle| z \in V\right\}$ if $y = 0$. Hence $(cd)f = c(df)$. Suppose that $c \neq 0 = d$. Then $((c(df))(x, y) = (df)(x, c^{-1}y) = 0$ if $y \neq 0$ and $= \sup\left\{f\left(x, c^{-1}z\right) \middle| z \in V\right\}$ if $y = 0$. Hence $(cd)f = c(df)$. Suppose that $c = 0 = d$. Then $\left(c(df)\right)(x, y) = 0$ if $y \neq 0$ and $= \sup\left\{(0f)(x,z) \middle| z \in V\right\} = (0f)(x, 0) = \sup\left\{f(x,z) \middle| z \in V\right\}$ if $y = 0$. Hence $(cd)f = c(df)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(ii) Suppose that $c \neq 0$. Then $(c(f+g))(x,y) = (f+g)(x, c^{-1}y) = \inf\{f(x, c^{-1}y_1), g(x, c^{-1}y_2)\}$ $= \inf\{(cf)(x,y_1), (cg)(x,y_2)\} = (cf+cg)(x,y)$ where $y = y_1 + y_2$, $f(x,y_1) > 0$, and $g(x,y_2) > 0$ by Proposition 1.10. Suppose that $c = 0$. Then $(0(f+g))(x,y) = 0$ if $y \neq 0$ and $= \sup\{(f+g)(x,z) | z \in V\} = \sup\{\inf\{f(x,z_1), g(x,z_2)\} | z \in V, z = z_1 + z_2\}$ if $y = 0$. Now $(0f + 0g)(x, y) = \inf\{(0f)(x,y_1), (0g)(x,y_2)\}$ where $y = y_1 + y_2$, $(0f)(x, y_1) > 0$, and $(0g)(x, y_2) > 0$. Thus $(0f + 0g)(x, y) = 0$ if $y_1 \neq 0$ or $y_2 \neq 0$ and $= \inf\{\sup\{f(x,z_1) | z_1 \in V\}, \sup\{g(x,z_2) | z_2 \in V\}\}$ if $y_1 = 0 = y_2$. Hence $0(f+g) = 0f + 0g$.

(iii) Suppose that $c + d \neq 0$ and $c \neq 0 \neq d$. Then $((c+d)f)(x,y) = f(x, (c+d)^{-1}y)$ and $(cf + df)(x, y) = \inf\{(cf)(x,y_1), (df)(x,y_2)\} = \inf\{f(x, c^{-1}y_1), f(x, d^{-1}y_2)\}$ where $y = y_1 + y_2$, $(cf)(x, y_1) > 0$, and $(df)(x, y_2) > 0$. Now $f(x, (c+d)^{-1}y) > 0$, $f(x, c^{-1}y_1) > 0$, and $f(x, d^{-1}y_2) > 0$ imply that $(c+d)^{-1}y = c^{-1}y_1 = d^{-1}y_2$ and so $y = c^{-1}(c+d)y_1$. Since $\exists y \in V$ such that $f(x, (c+d)^{-1}y) > 0$, $y = y_1 + y_2$, $(c+d)^{-1}y = c^{-1}y_1 = d^{-1}y_2$, we have that $(c+d)f = cf + df$. (Take $y_1 = c(c+d)^{-1}y$, $y_2 = d(c+d)^{-1}y$.) Suppose that $c = 0 \neq d$. Then $(c+d)f = df$. Now $(0f + df)(x, y) = \inf\{(0f)(x,y), (df)(x,y)\} = (df)(x, y)$ where $(0f + df)(x, y) > 0$. Thus $(c+d)f = cf + df$. Suppose that $c + d = 0$ and $c \neq 0$. Then $d = -c$. Now $(cf + -cf)(x, y) = \inf\{(cf)(x,y_1), (-cf)(x,y_2)\}$ $= \inf\{f(x, c^{-1}y_1), f(x, (-c)^{-1}y_2)\}$ where $y = y_1 + y_2$, $(cf)(x, y_1) > 0$, and $(-cf)(x, y_2) > 0$ and so $c^{-1}y_1 = (-c^{-1})y_2$, $y_1 = -y_2$, $y = 0$. Now $(0f)(x, 0) = f(x, z) > 0$ for some $z \in U$. Thus $z = c^{-1}y_1$. Hence $0f = cf + -cf$. Suppose that $c = 0 = d$. Then $(c+d)f = 0f$. Now $(0f + 0f)(x, y) = \inf\{(0f)(x,y_1), (0f)(x,y_2)\}$ where $(0f + 0f)(x, y) > 0$ and $y = y_1 + y_2$. Thus $y_1 = y_2 = 0 = y$. Hence $(0f + 0f)(x, y) = (0f)(x, y)$. Thus $(c+d)f = cf + df$.

(iv) Immediate.

COROLLARY 1.12. *$\mathcal{FL}_t(U, V)$, $t \neq 0$, is a vector space over F. In fact $\mathcal{FL}_t(U,V) \simeq \mathcal{L}(U,V)$, the vector space of all crisp linear transformations of U into V.*

Proof. $f \mapsto T$ is an isomorphism of $\mathcal{FL}_t(U,V)$ onto $\mathcal{L}(U,V)$ where $f(x, y) = t$ iff $T(x) = y$.

2. Fuzzy Norms. We now let F denote R, the field of real numbers, or C, the field of complex numbers. Let U be a linear space over F.

DEFINITION 2.1. *Let N be fuzzy subset of $U \times R$. The N is called a fuzzy norm on U if and only if $\forall x, u \in U$, $\forall c \in F$,*

(i) *$\forall t \in R$ with $t \leq 0$, $N(x, t) = 0$;*

(ii) *$\forall t \in R$, $t > 0$, $N(x, t) = 1$ if and only if $x = 0$;*

(iii) *$\forall t \in R$, $t > 0$, $N(cx, t) = N(x, t/|c|)$ if $c \neq 0$ and $N(cx, t) = 1$ if $c = 0$;*

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(iv) $\forall t, s \in R$, $N(x + u, t + s) \geq \min \{N(x,t), N(u,s)\}$;

(v) $N(x, -)$ *is a left continuous and nondecreasing function of* R *such that* $\lim_{t\to\infty} N(x,t) = 1$. *The pair* (U, N) *will be referred to as a fuzzy normed linear space.*

PROPOSITION 2.2. *Let* N *be a fuzzy norm on* U. *Then* $\forall x \in U, \forall t \in R, N(x,t) = N(-x,t)$.

Proof. $N(-x,t) = N(-1x,t) = N(x,t/|-1|)$ if $t > 0$ and $N(-x,t) = 0 = N(x,t)$ *if* $t \leq 0$.

PROPOSITION 2.3. *Let* N *be a fuzzy norm on* U. *Define the fuzzy subset* ρ *of* $U \times U \times R$ *by* $\forall (x,u,t) \in U \times U \times R, \rho(x,u,t) = N(x-u,t)$. *Then* ρ *is a fuzzy metric on* U *in the sense of (Kramosil & Michalek 1975, Definition 7, p. 341)*

Proof. (i) Suppose that $t \leq 0$. Then $\rho(x, u, t) = N(x - u, t) = 0$.

(ii) $\forall t > 0$, $\rho(x, u, t) = 1$ iff $\forall t > 0$, $N(x - u, t) = 1$ iff $x - u = 0$ iff $x = u$.

(iii) $\rho(x, u, t) = N(x - u, t) = N(u - x, t) = \rho(u, x, t)$.

(iv) $\rho(x, u, t + s) = N(x - u, t + s) = N(x - z + z - u, t + s) \geq \min \{N(x-z,t)N(z-u,s)\} = \min \{\rho(x,z,t), \rho(z,u,s)\}$

(v) $\rho(x, u, -) = N(x - u, -)$ and so $\rho(x, u, -)$ is a left continuous and nondecreasing function of R such that $\lim_{t\to\infty} \rho(x,u,t) = 1$.

DEFINITION 2.4. *Let* (U, N_U) *and* (V, N_V) *be fuzzy normed linear sapces over* F. *Let* $f \in \mathcal{FH}(U, V)$ *and* $u \in U$. *Then* f *is* ***continuous*** *at* u *if and only if* $\forall \varepsilon > 0$, $\exists \delta > 0$ *such that* $\forall x \in U$, $N_V(y - v, \varepsilon) \geq N_U(x - u, \delta)$ *where* $f(u,v) > 0$ *and* $f(x,y) > 0$. f *is said to be* ***continuous*** *on* U *if and only if* f *is continuous at every point on* U.

DEFINITION 2.5. *Let* (U, N_U) *and* (V, N_V) *be fuzzy normed linear spaces over* F. *Let* $f \in \mathcal{FH}(U, V)$. *Then* f *is* ***bounded*** *on* U *if and only if* $\exists$ *positive real number* M *such that* $\forall x \in U$, $\forall t \in R$, $N(y, t) \geq N(x, t/M)$ *where* $f(x, y) > 0$.

THEOREM 2.6. *Let* (U, N_U) *and* (V, N_V) *be fuzzy normed linear spaces over* F. *Let* $f \in \mathcal{FH}(U, V)$.

(i) $\exists u \in U$ *such that* f *is continuous at* u *if and only if* f *is continuous on* U;

(ii) f *is continuous on* U *if and only if* f *is bounded on* U.

Proof. (i) Suppose that f is continuous at u. Let $x_1 \in U$. Let $\varepsilon > 0$. Then $\exists \delta > 0$ such that $N(y - v, \varepsilon) \geq N((x - u, \delta)$ where $f(u, v) > 0$ and $f(x, y) > 0$. Thus $N(y' - v, \varepsilon) \geq N((x + u - x_1) - u, \delta)$ where $f(x + u - x_1, y') > 0$. Thus $N(y - y_1, \in) \geq N(x - x_1, \delta)$ where $f(x_1, y_1) > 0$ since $y - y_1 = y' - v$. Hence f is continuous at x_1.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(ii) Suppose that f is bounded on U. Let $\varepsilon > 0$. $\exists M > 0$ such that $N(y, \varepsilon) \geq N(x, \varepsilon/M)$ where $f(x, y) > 0$. *i.e.*, $N(y-0, \varepsilon) \geq N(x-0, \delta)$ where $\delta = \varepsilon/M$. Thus f is continuous at 0. Conversely, suppose that f is continuous. Then $\exists \delta > 0$ such that $N(y, 1) \geq N(x, \delta)$ where $f(x, y) > 0$ since f is continuous at 0. Suppose that $x \neq 0$. Let $t > 0$. Set $u = x/t$ and $v = y/t$. Then $N(y, t) = N(tv, t) = N(v, 1) \geq N(u, \delta) = N(x/t, \delta) = N(x, t/M)$ where $f(u, v) > 0$ and $M = 1/\delta$. Suppose that $t \leq 0$. Then $N(y, t) = 0 = N(x, t\delta)$ where $f(x, y) > 0$. Suppose that $x = 0$. Let $t \in R$. Then $N(0_U, t) = N(0_V, t) = N(0_V, t\delta) = 1$ if $t > 0$ and $N(0_V, t) = N(0_U, t) = N(0_U, t\delta) = 0$ if $t \leq 0$.

Note that $\forall x \in U$, $\exists t \in R$ such that $N(x, t) > 0$ since $\lim_{t \to \infty} N(x, t) = 1$.

PROPOSITION 2.7. *Let* (U, N_U) and (V, N_V) *be fuzzy normed linear spaces over F. Suppose that* $N(x, t) > 0$ $\forall t > 0$. *implies* $x = 0$. *Define* $\| \ \|$ *on* U *by* $\forall x \in U$, $\|x\| = \inf\{t / N(x, t) > 0\}$. *Then* $\| \ \|$ *is a norm on U.*

Proof. $\forall x \in U$, $\|x\| \geq 0$ since $N(x, t) > 0$. Now $\|x\| = 0$ iff $\inf\{t | N(x, t) > 0\} = 0$ iff $\forall t \in R$ such that $t > 0$, $N(x, t) > 0$ iff $x = 0$. Let $c \in F$, $c \neq 0$, and $x \in U$. Then $\|cx\| = \inf\{s | N(cx, s) > 0\}$ $= \inf\{s | N(x, s/|c|) > 0\} = \inf\{|c|t | N(x, t) > 0\} = |c| \inf\{t | N(x, t) > 0\} = |c| \ \|x\|$. Suppose that $c = 0$. Then $\|cx\| = \|0\| = \inf\{t | N(0, t) > 0\} = \inf\{t | t > 0\} = 0 = 0\|x\| = |c| \ \|x\|$. Now $\|x\| + \|y\| = \inf\{t | (x, t) > 0\}$ $+ \inf\{s | N(x, s) > 0\} = \inf\{t + s | N(x, t) > 0, N(y, s) > 0\}$ (since $s, t > 0$) $\geq \inf\{t + s | N(x + y, t + s) > 0\}$ (since $N(x, t) > 0, N(y, s) > 0$ implies $N(x + y, t + s) > 0$) $= \|x + y\|$.

References

Biswas, R. (1991) : Fuzzy inner product spaces and fuzzy norm functions, *Inf. Sci.* **53**, 185.

El-Abyad, A.M. and El-Hamouly, H.M. (1991) : Fuzzy inner product spaces, *Fuzzy Sets and Systems* **44**, 309.

Katsaras, A.K. (1981) : Fuzzy topological vector spaces, *Fuzzy Sets and Systems*, **6**, 85.

Katsaras, A.K. (1984) : Fuzzy topological vector spaces II, *Fuzzy Sets and Systems* **12**, 143.

Katsaras, A.K. and Liu, D. B. (1977) : Fuzzy vector spaces and fuzzy topological vector spaces, *J. Math. Anal.Appl.* **58**, 135.

Khan, Khodadad A. F. M. and Ahsanullah, T. M. G. (1993) : Some properties of fuzzy neighbourhood normed linear spaces, *Fuzzy Sets and Systems* **54**, 333.

Klir, G.J. and Folger, T.A. (1988) : *Fuzzy Sets, Uncertainty, and Information*, Prentice Hall, Englewood Cliffs, New Jersey.

Kramosil, I. and Michalek, J. (1975) : Fuzzy metrics and statistical metric spaces, *Kybernetika* **11**, 336.

Krishna, S.V. and Sarma, K.K.M. (1991) : Fuzzy topological vector spaces–topological generation and normability, *Fuzzy Sets and System* **41**, 89.

Malik, D.S. and Mordeson, J.N. (1992) : Fuzzy homomorphisms of rings, *Fuzzy Sets and System* **46**, 139

Morsi, N.N. (1988) : On fuzzy pseudo-normed vector spaces, *Fuzzy Sets and System* **27**, 351.

Petrich, M. (1973) : *Introduction to Semigroups*, Charles E. Merril Publishing Co., A Bell and Howell Co., Columbus, Ohio.

Wu Congxin and Fang Ginxuan (1984) : Fuzzy generalization of Kolmogoroff's theorem, *J. Harbin Inst. Technology* 1.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Wu Cong-Xin and Ma Ming (1990) : Fuzzy norms, probabilistic norms, and fuzzy metrics, *Fuzzy Sets and System* 36, 137

DEPARTMENT OF MATHEMATICS/
COMPUTER SCIENCE
CREIGHTON UNIVERSITY
OMAHA, NEBRASKA 68178
U. S. A.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc. **86**, 437-444 (1994)

ON PRIME RADICAL OF Γ-SEMIGROUP

T. K. DUTTA AND N. C. ADHIKARI

(Received 21 May 1993; revision received 25 August 1993)

Abstract : In this paper we introduce the notions of prime ideal and prime radical in S and Γ where S is a Γ-semigroup and study them. We also obtain the relations among the prime radicals of S, Γ, M, and N *where* M *and* N are respectively the left operator semigroup and the right operator semigroup of the Γ-semigroup S.

§ 1. Introduction : In the year 1986 Sen and Saha introduced the notion of Γ-semigroup. Many classical notions of semigroup such as, ideals, Rees matrix representation, regular semigroup, inverse semigroup and orthodox semigroup have been generalised to Γ-semigroup (Sen and Saha 1990, Saha 1987, 1988 Saha and Seth 1989). In (Dutta and Adhikari 1994) we consider both sided Γ-semigroup S and construct a semigroup Em(S) with zero in which S is embedded as a Γ-semigroup and study the relations between the properties of S and those of Em(S). In (Dutta and Adhikari 1993) we introduce the notions of left unity and right unity in a Γ-semigroup S. We also define left operator semigroup and right operator semigroup of a Γ-semigroup S and study the relations among the set of ideals of S, Γ, M and N. Lastly we define Γ-group and study it.

In this paper we define prime ideal and prime radical in S and Γ where S is a both sided Γ-semigroup. We observe that there is an inclusion preserving bijection between the set of prime ideals of S and the set of prime ideals of the left operator semigroup M (right operator semigroup N) of the Γ-semigroup S and similar results also hold for Γ. We also define prime radical in S and Γ. We obtain relations among the prime radicals of S, Γ, M and N.

§ 2. Preliminaries : Let S and Γ be two nonempty sets. Now S is called a Γ-semigroup if there exist mappings from $S \times \Gamma \times S$ to S, written as $(a, \alpha, \beta) \to a\alpha b$ and from $\Gamma \times S \times \Gamma$ to Γ, written as $(\alpha, a, \beta) \to \alpha a \beta$ ...(i) satisfying the associative law $(a\alpha b)\beta c = a(\alpha b \beta)c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. A Γ-semigroup S is called a both sided Γ-semigrop if it further satisfies the following associative law $\alpha a\ (\beta\, b\gamma) = \alpha(a\beta b)\gamma = (\alpha a\beta)b\gamma$ for all $a, b, c \in S$ and $\alpha, \beta, \gamma \in \Gamma$. As a natural example of both sided Γ-semigroup we consider the following example.

EXAMPLE. Let A and B be two nonempty sets. Let S be the set of all mappings from A to B and Γ be the set of all mapping from B to A. Then S is a both sided Γ-semigroup where $a\alpha b$ and $a\alpha b$ and $\alpha a \beta$ $(a, b \in S, \alpha, \beta \in \Gamma)$ denote the usual composition of mappings.

The followings are also the examples of both sided Γ-semigroup.

EXAMPLE . Let S be the set of all 2×3 matrices over the set of positive integers and Γ be the set of all 3×2 matrices over the same set. Then S is a both sided Γ-semigroup where $\alpha a \beta$ $(a, b \in S, \alpha, \beta \in \Gamma)$ denote the usual product of matrices.

EXAMPLE. Let S be the set of all integers of the form $4n + 1$ where n is an integer and Γ denote the set of all integers of the form $4n + 3$. If $a\alpha b$ is $a + \alpha + b$, $\alpha a \beta$ is $\alpha + a + \beta$ (usual sum of the integers) for all $a, b \in S$ and $\alpha, \beta \in \Gamma$ then S is a both sided Γ-semigroup.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

EXAMPLE. Let (X, τ_1) and (Y, τ_2) be two topological spaces. Let S be the set of all continuous maps from (X, τ_1) to (Y, τ_2) and Γ be the set of all continuous maps from (Y, τ_2) to (X, τ_1). Then S is a both sided Γ-semigroup where $f\alpha g$ and $\alpha f\beta$ denote the usual composition of maps $(f, g \in S, \alpha, \beta \in \Gamma)$.

Throughout this paper we consider only both sided Γ-semigroup and call it simply Γ-semigroup. A nonempty subset I of S is called a left (right) ideal of S if $S\Gamma I \subseteq I$ (*resp.* $I\Gamma S \subseteq I$). A nonempty subset I of S is called a two sided ideal or simply an ideal of S if it is a left ideal as well as a right ideal of S. The ideals of Γ are defined analogously. For each element a of S, the smallest right ideal containing a is called the principal right ideal of S generated by a and is denoted by $|a>$. Similarly we define $<a|$ and $<a>$, the principal left and two sided ideals of S generated by a. The principal ideals of Γ are defined analogously.

Let S be a Γ-semigroup. We define a relation ζ on $S \times \Gamma$ as follows : $(x, \alpha)\ \zeta\ (y, \beta)$ if and only if $x\alpha s = y\beta s$ for all $s \in S$ and $\gamma x\alpha = \gamma y\beta$ for all $\gamma \in \Gamma$. Obviously ζ is an equivalence relation. Let $[x, \alpha]$ denote the equivalence class containing (x, α). Let $M = \{[x, \alpha] : x \in S$ and $\alpha \in \Gamma\}$. We define a multiplication in M as follows :

$[x, \alpha]\,[y, \beta] = [x\alpha y, \beta] = [x, \alpha y\beta]$. M is a semigroup with respect to this multiplication. This semigrop M is called the left operator semigroup of the Γ-semigroup S. Similarly we can define the right operator semigroup $N = \{[\alpha, a] : \alpha \in \Gamma$ and $a \in S\}$ of the Γ-semigroup S where the multiplication in N is defined by $[\alpha, a]\,[\beta, b] = [\alpha a\beta, b] = [\alpha, a\beta b]$. For subsets $R \subseteq S, \phi \subseteq \Gamma$, we denote by $[R, \phi]$ the set of all elements $[a_i, \alpha_i]$ in M where $a_i \in R$ and $\alpha_i \in \phi$. Thus in particular $M = [S, \Gamma]$ and $N = [\Gamma, S]$.

The set S is a left M-set and a right N-set under the definitions $[a, \alpha]s = a\alpha s$ and $s[\beta, b] = s\beta b$. Furthermore, $([a, \alpha]s)[\beta, b] = [a, \alpha](s[\beta, b])$. So S is a (M, N)-biset. Similarly Γ is a (N, M)-biset (Clifford and Preston 1961).

For $P \subseteq M(N)$ we define $P^+ = \{x \in S : [x, \alpha] \in P$ for all $\alpha \in \Gamma\}$, ${}^+P = \{\gamma \in \Gamma : [x, \gamma] \in P$ for all $x \in S\}$, (resp. $P^* = \{x \in S : [\alpha, x] \in P$ for all $\alpha \in \Gamma\}$, ${}^*P = \{\gamma \in \Gamma : [\gamma, x] \in P$ for all $x \in S\}$). Then it follows that if P is an ideal of M (N) then $+P$ (P^*) is an ideal of S and $P + ({}^*P)$ is an ideal of Γ [4].

Similarly for $Q \subseteq S(\Gamma)$ we define $Q^{+\prime} = \{[x, \alpha] \in M : x\alpha s \in Q$ for all $s \in S\}$ and $Q^{*\prime} = \{[\alpha, x] \in N : s\alpha x \in Q$ for all $s \in S\}$ (resp. ${}^{+\prime}Q = \{[x, \alpha] \in M : \gamma x\alpha \in Q$ for all $\gamma \in \Gamma$ and ${}^{*\prime}Q = \{[\alpha, x] \in N : \alpha x\gamma \in Q$ for all $\gamma \in \Gamma\}$). It follows that if Q is an ideal of $S(\Gamma)$ then $Q^{+\prime}$ and ${}^{+\prime}Q$ are ideals of M (resp. $Q^{*\prime}$ and ${}^{*\prime}Q$ are ideals of N) (Dutta and Adhikari 1993).

§ 3. Prime Ideal :

DEFINITION 3.1. *Let S be a Γ-semigroup. An ideal $P (\neq S)$ of S is said to be prime if, for any two ideals A and B of S, $A\Gamma B \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$. The prime ideals of Γ is defined analogously.*

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

EXAMPLE 3.2. Let S be the set of all 1×2 matrices over GF_2 and Γ be the set of all 2×1 matrices over GF_2. Then S is a both sided Γ-semigroup where $a\gamma b$ and $\alpha a \beta$ denote the usual matrix prodct ($a, b \in S$ and $\alpha, \beta, \gamma \in \Gamma$). It can be shown that $\{(0, 0)\}$ is a prime ideal of S.

EXAMPLE 3.3. Let S be a semigroup. The S may be considered as a Γ-semigroup where $\Gamma = S$ and $a\alpha b$ and $\alpha a \beta$ denote the usual product of the elements in S then every prime ideal of the semigroup S is a prime ideal of S where S is a Γ-semigroup.

THEOREM 3.4. *Let S be a Γ-semigroup. For an ideal P ($\neq S$) of S, the following conditions are equivalent.*

(i) *If A and B are ideals of S such that $A\Gamma B \subseteq P$ then either $A \subseteq P$ or $B \subseteq P$.*

(ii) *If $< a >$, $< b >$ are principal ideals of S such that $< a > \Gamma < b > \subseteq P$ then either $a \in P$ or $b \in P$.*

(iii) *If $a\Gamma S\Gamma b \subseteq P$ then $a \in P$ or $b \in P$.*

(iv) *If I_1 and I_2 are two right ideals of S such that $I_1\Gamma I_2 \subseteq P$ then either $I_1 \subseteq P$ or $I_2 \subseteq P$.*

(v) *If J_1 and J_2 are two left ideals of S such that $J_1\Gamma J_2 \subseteq P$ then either $J_1 \subseteq P$ or $J_2 \subseteq P$.*

Proof. (i) $\Rightarrow$ (ii). Obvious.

(ii) $\Rightarrow$ (i). Suppose A and B are ideals of S such that $A\Gamma B \subseteq P$ but $A \subseteq P$. Then there exists $a \in A$ such that $a \notin P$. Now for any $b \in B$ we have $< a > \Gamma < b > \subseteq A\Gamma B \subseteq P$, whence $b \in P$ (by (i)). Thus $B \subseteq P$.

(i) $\Rightarrow$ (iii). Suppose $a\Gamma S\Gamma b \subseteq P$. Then $< a > \Gamma < a > \Gamma < b > \Gamma < b > \subseteq P$. Since $< a > \Gamma < a >$, $< b > \Gamma < b >$ are ideals of S, so by (i) we have either $< a > \Gamma < a > \subseteq P$ or $< b > \Gamma < b > \subseteq P$. By repeated use of (i) we get $a \in < a > \subseteq P$. or $b \subset \in < b > \subseteq p$.

(iii) $\Rightarrow$ (iv) Let $I_1\Gamma I_2 \subseteq P$. Let $I_1 \subset P$, then there exists an element $a_1 \in I_1$ such that $a_1 \notin P$. Then for every $a_2 \in I_2$ we have $a_1\Gamma S\Gamma a_2 \subseteq I_1\Gamma_2 \subseteq P$. Hence from (iii) $a_2 \in P$. Thus $I_2 \subseteq P$. Similarly (iii) $\Rightarrow$ (v)

The proof is completed by observing that (i) is implied either by (iv) or by (v).

The following results are easy to prove.

PROPOSITION 3.5. *Let S be a Γ-semigroup. If I is an ideal of S and P a prime ideal of S, then $I \cap P$ is a prime ideal of I, considering I as a Γ-semigroup.*

PROPOSITION 3.6. *Let S be a Γ-semigroup. If P is a prime ideal of S and a is an element of S such that $S\Gamma a\Gamma S \subseteq P$ then $a \in P$.*

DEFINITION 3.7. *Let S be a Γ-semigroup. A subset H of S is said to be an m-System of S if and only if $c \in H$, $d \in H$ imply that there exist elements $p \in S$ and $\alpha_1, \alpha_2 \in \Gamma$ such that $c\alpha_1 p\alpha_2 d \in H$. The empty set is to be considered as an m-System of S.*

PROPOSITION 3.8 *Let S be a Γ-semigroup. An ideal P ($\neq S$) of S is prime if and only if its complement P^c is an m-System.*

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

PROPOSITION 3.9. *Let S be a Γ-semigroup. Let A be an ideal disjoint from an m-System H of S. Then there exists an m-System $T \supseteq H$ which is maximal in the class of m-System disjoint from A.*

Proof. Zorn's lemma is applied to the class of m-System disjoint from A.

PROPOSITION 3.10. *Let S be a Γ-semigroup. Let A be an ideal of S disjoint from an m-system H of S. Then there exists an ideal $P \supseteq A$ which is maximal in the class of ideals containing A and disjoint from H. Moreover, P is prime.*

We recall the following definitions.

Let S be a Γ-semigroup with the left operator semigroup M. For $P \subseteq M$ and $Q \subseteq S$ we define $P^+ = \{x \in S : [x, \alpha] \in P \text{ for all } \alpha \in \Gamma\}$ and $Q^{+'} = \{[x, \alpha] \in M : x\alpha s \in Q \text{ for all } s \in S\}$.

LEMMA 3.11. *If P is a prime ideal of M then P^+ is a prime ideal of S.*

Proof. Suppose $A\Gamma B \subseteq P^+$ where A and B are two ideals of S. Then $[A, \Gamma][B, \Gamma] = [A\Gamma B, \Gamma] \subseteq P$. Now since A and B are ideals of S, $[A, \Gamma]$ and $[B, \Gamma]$ are ideals of M. Therefore by the primeness of P, either $[A, \Gamma] \subseteq P$ or $[B, \Gamma] \subseteq P$. This means either $A \subseteq P^+$ or $B \subseteq P^+$. Hence P^+ is a prime ideal of S.

LEMMA 3.12. *If Q is a prime ideal of S then $Q^{+'}$ is a prime ideal of M.*

Proof. Suppose $AB \subseteq Q^{+'}$ where A and B are ideals of M. Then $AMB \subseteq AB \subseteq Q^{+'}$. Hence $AS\Gamma BS \subseteq Q^{+'}S \subseteq Q$. Now since AS and BS are ideals of S, by the primeness of Q, it follows that either $AS \subseteq Q$ or $BS \subseteq Q$. This means that either $A \subseteq Q^{+'}$ or $B \subseteq Q^{+'}$. Hence $Q^{+'}$ is a prime ideal of M.

THEOREM 3.13. *Let S be a Γ-semigroup and M be its left operator semigroup. Then there exists an inclusion preserving bijection $Q \to Q^{+'}$ between the set of all prime ideals of S and the set of all prime ideals of M.*

Proof. Let Q be a prime ideal of S. Then, by lemma 3.12, $Q^{+'}$ is a prime ideal of M. Now $\left(Q^{+'}\right)^+ = \left\{x \in S : [x, \alpha] \in Q^{+'} \text{ for all } \alpha \in \Gamma\right\} = \{x \in S : x\alpha a \in Q \text{ for all } \alpha \in \Gamma \text{ and for all } a \in S\}$. As $Q\Gamma S \subseteq Q$, $Q \subseteq \left(Q^{+'}\right)^+$. Again $\left(Q^{+'}\right)^+ \Gamma S \subseteq Q$. Now by the primeness of Q, $\left(Q^{+'}\right)^+ \subseteq Q$ (Since $S \subseteq Q$). Thus $Q = \left(Q^{+'}\right)^+$. Let U be a prime ideal of M. Then we have

$\left(U^+\right)^{+'} = \left\{[x, \alpha] \in M : [x, \alpha]s \in U^+ \text{ for all } s \in S\right\} = \left\{[x, \alpha] \in M : [x\alpha s, \gamma] \in U \text{ for all } s \in S \text{ and for all } \gamma \in \Gamma\right\}$

As $UM \subseteq U$, $U \subseteq \left(U^+\right)^{+'}$. Again as $\left(U^+\right)^{+'} M \subseteq U$, by primeness of U, $\left(U^+\right)^{+'} \subseteq U$ (since $M \subseteq U$). Thus $U = \left(U^+\right)^{+'}$. Now it follows that the mapping $Q \to Q^{+'}$ is an inclusion preserving bijection between the set of prime ideals of S and the set of prime ideals of M. Similarly we can prove the following theorems.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

THEOREM 3.14. *Let S be a Γ-semigroup with right operator semigroup N. Then there is an inclusion preserving bijection between the set of all prime ideals of S and the set of all prime ideals of N via the mapping* $Q \to Q^{*'}$ *where Q is a prime ideal of S.*

THEOREM 3.15. *Let S be a Γ-semigroup with the left operator semigroup M and right operator semigroup N. Then there is an inclusion preserving bijection between the set of all prime ideals of Γ and the set of all prime ideals of M (N) via the mapping* $V \to^{+'} V\left(resp. V \to^{*'} V\right)$ *where V is a prime ideal of Γ. Combining Theorems 3.14 and 3.15 we have.*

THEOREM 3.16. *Let S be a Γ-semigroup. Then there exists an inclusion preserving bijection between the set of all prime ideals of S and the set of all prime ideals of Γ via the mapping* $A \to^{+} \left(A^{+'}\right) or\, A \to^{*} \left(A^{*'}\right)$.

§ 4. Prime Radical :

DEFINITION 4.1. *Let S be a Γ-semigroup. The intersection* K_S *of all ideals of S, if it is nonempty, is called the Kernel of S. The Kernel of Γ is defined analogously.*

DEFINITION 4.2. [Definition 3.1, 4] *Let S be a Γ-semigroup. M and N be the left operator semigroup and the right operator semigroup of the Γ-semigroup S. If there exists an element* $[e,\delta] \in M$ *([δ, e] ∈ N] such that eδx = x (resp. xδe = x) for all x ∈ S then S is said to have the left unity [e, δ] (resp. right unity [δ, e]). Similarly we can define the left unity and right unity of Γ. A Γ-semigroup S is said to be a Γ-semigroup with unities if S has left and right unity which are also right and left unity of Γ respectively and vice-versa.*

We assume throughout the balance of this paper that S is a Γ-semigroup with unities and also S has Kernel K_S.

THEOREM 4.3. [Theorem 4.3,4] *Let S be a Γ-semigroup with the left operator semigroup M. Then there exists an inclusion preserving bijection* $Q \to Q^{+'}$ *between the set of all right ideals of S and the set of all right ideals of M.*

LEEMA 4.4. *Let S be a Γ-semigroup and M be its left operator semigroup. Kernel* K_S *of S exists if and only if Kernel* K_M *of M exists and* $K_S^{+'} = K_M, K_M^{+} = K_S$.

Proof. Let the Kernel K_S of S exists. Since K_S is an ideal of S, by the Proposition 4.2 (Dutta and Adhikari 1993, $K_S^{+'}$ is an ideal of M. Let Q be an ideal of M. By Proposition 4.1 (Dutta and Adhikari 1993), Q^{+} is an ideal of S. Hence $K_S \subseteq Q^{+}$. Again, since S is a Γ-semigroup with unities and +' is inclusion preserving bijection between the set of all ideals of S and the set of all ideals of M, hence $K_S^{+'} \subseteq \left(Q^{+}\right)^{+'} = Q$. Thus $K_S^{+'}$ is the smallest ideals of M and hence $K_M = K_S^{+'}$. Similarly we can show that. $K_M^{+} = K_S$.

DEFINITION 4.5. *Let S be a Γ-semigroup. The prime radical P(S) of S is the intersection of all prime ideals of S.*

Remark. $P(S)$ is nonempty since $K_s \subseteq P(S)$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

THEOREM 4.6. *If $P(M)$ is the prime radical of the left operator semigroup M of a Γ-semigroup S, then* $P(S) = P(M)^+$.

Proof. Let Q be a prime ideal of S. Then by Lemma 3.12, it follows that $Q^{+'} = P$ is a prime ideal of M. Hence $P^+ = \left(Q^{+'}\right)^+ = Q$. Let $\mathscr{C}$ and $\mathscr{D}$ be the collections of all prime ideals of S and those of M respectively. Now, from Lemma 3.11, it follows that $P \in \mathscr{D}$ implies $P^+ \in \mathscr{C}$. Hence $P(S) = \bigcap_{Q\in\mathscr{C}} Q \subseteq \bigcap_{P\in\mathscr{D}} P^+ = \left(\bigcap_{P\in\mathscr{D}} P\right)^+ = P(M)^+$. Again, $P(S) = \bigcap_{Q\in\mathscr{C}} Q = \bigcap_{P\in\mathscr{D}'} P^+$ ($\mathscr{D}'$ is a subcollection of $\mathscr{D}$) $= \left(\bigcap_{P\in\mathscr{D}'} P^+\right) \supseteq \left(\bigcap_{P\in\mathscr{D}} P\right)^+ = P(M)^+$. Thus $P(S) = P(M)^+$.

THEOREM 4.7. *Let S be a Γ-semigroup. If $P(M)$ is the prime radical of the left operator semigroup M of the Γ-semigroup S, then $P(M) = P(S)^{+'}$.*

Proof. From Theorem 4.6, it follows that $P(S) = P(M)^+$.

Hence $P(S)^{+'} = \left(P(M)^+\right)^{+'} = P(M)$.

The following result is a consequence of the Theorem 4.6 and 4.7.

LEMMA 4.8. *If P (M) is the prime radical of the left operator semigroup M of a Γ-semigroup S, then $P(M) = P(M)^{++'}$ and $P(S) = P(S)^{+'+}$.*

THEOREM 4.9. *Let S be Γ-semigroup and A an ideal of S. Then the Kernel of A exists and is equal to K_S.*

Proof. Obviously $K_S \subseteq A$. Let I be an ideal of A. Then $K_S \subseteq I$. For, if $K_S \subseteq I$, then there exists an element $a \in K_S$ such that $a \notin I$. Since $A\Gamma I\Gamma A \subseteq I$, so $a \notin A\Gamma I\Gamma A$. Again $A\Gamma I\Gamma A$ is an ideal of S and $K_S \subseteq A\Gamma I\Gamma A$, so $a \notin A\Gamma I\Gamma A$ implies that $a \notin K_S$, a contradiction. Hence K_S is contained in all ideals of A; inotherwords K_S is the smallest ideal of A, so K_A exists and $K_A = K_S$.

LEMMA 4.10. *Let S be a Γ-semigroup. Then $P(S) = \{s \in S$: every m-System of S which contains is contains an element of $K_S\}$.*

Proof. $P' = \{s \in S$: every m-System of S which contains s contains an element of $K_S\}$. Let $s \notin P(S)$. Then there is a prime ideal P of S such that $s \notin P$. Then P^c is an m-System of S containg s but $P^c \cap K_S = \phi$, since $K_S \subseteq P$. Hence $s \notin P'$. Next, let $s \in P'$, then there is an m-System H containing s such that $H \cap K_S = \phi$. Now by the Proposition 3.10 there exists a prime ideal Q of S such that $Q \cap H = \phi$. Since $s \in H$, $s \notin Q$. Hence $s \notin P(S)$. This completes the proof.

THEOREM 4.11. *Let S be a Γ-semigroup. If I is an ideal of S then $P(I) = I \cap P(S)$ where $P(I)$ denotes the prime radical of I considering I as a Γ-semigroup.*

Proof. Let $\mathscr{C}$ be the collection of all prime ideals of S and $\mathscr{D}$ be the collection of all prime ideals of I. From Proposition 3.5, it follows that if $P \in \mathscr{C}$ then $P \cap I \in \mathscr{D}$. So

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

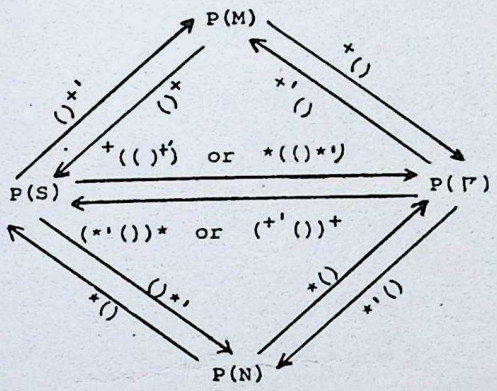
Digitized by Arya Samaj Foundation Chennai and eGangotri

$P(I) = \bigcap_{P \in \mathcal{D}} P \subseteq \bigcap_{Q \in \mathcal{C}} (Q \cap I) = \bigcap_{Q \in \mathcal{C}} Q \cap I = P(S) \cap I$. Again, every m-System of I is certainly an m-System of S. Let $a \notin P(I)$. Then there exists an m-System H of I containing a such that $H \cap K_I = \phi$.. Now H is an m-System of S and $K_I = K_S$. So there exists an m-System H of S containing a such that $H \cap K_S = \phi$. So $a \notin P(S)$. Hence $I \cap P(S) \subseteq P(S) \subseteq P(I)$. Thus $P(I) = I \cap P(S)$.

THEOREM 4.12. *Let S be a Γ-semigroup. If I is an ideal of S then* $P(I)^{+'} = I^{+'} \cap P(M)$.

Proof. From Theorem 4.11, it follows that $P(I)^{+'} = \left(I \cap P(S)\right)^{+'} = I^{+'} \cap P(S)^{+'} = I^{+'} \cap P(M)$ (using Theorem 4.7).

Analogously we obtain the relation between the prime radical $P(S)$ of S and the prime radical $P(N)$ of the right operator semigroup N of Γ-semigroup S and the relation between the prime radical $P(\Gamma)$ of Γ (on assuming that K_Γ exists) and the prime radicals of M (and N). We show these relationships in the following commutative diagram.



Acknowledgement : We are thankful to the learned referee for his valuable suggestions.

References

Clifford, A.H. and Preston, G. B (1961) : "The algebraic theory of semigroups" I and II, *Amer. Math. Soc., Providence, R. I.* (1961) and (1967).

Dutta, T.K. and Chatterjee, T.K. (1988) : Green's equivalences on Γ-semigroup. *Bull. Cal. Math. Soc.* **80**, 30.

Dutta, T.K. and Adhikari, N. C. (1994) : Embedding a Γ-semigroup in a semigroup with zero. *Bull. Cal. Math. Soc.* 86, 1.

Dutta, T. K. and Adhikari, N.C. (1993) : On Γ-semigroup with the right and left unities. *Soochow Jr. of Math.* **19**, 4, 461.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Sen,M. K. (1981) : *On Γ-semigroup. Proceedings of the international conference on Algebra and its application.* Decker Publication, New York, 301

Sen, M.K. and Saha, N. K. (1986) : On Γ-semigroup I. *Bull. Cal. Math. Soc.* **78,** 380.

Sen, K. K. and Saha, N. K. (1990) : Orthodox Γ-semigroup. *Inter. Jr. Math. and Math. Sc., 3, 527.*

Saha, N. K. (1987) : On Γ-semigroup -II *Bull. Cal. Math. Soc.* **79** (1987), 331.

Saha, N. K. (1988) : On Γ-semigroup-III *Bull. Cal. Math. Soc.* **80,** 1

Saha, N. K. and Seth, A. (1994) Γ Inverse Γ-semigroup. Accepted for publication in *Journal of Pure Maths. Calcutta University.*

Seth, A. (1989) : Rees's Theorem for Γ-semigroup *Bull. Cal. Math. Soc.* **81.,** 217

DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF CALCUTTA,
35, BALLYGUNGE CIRCULAR ROAD,
CALCUTTA-700 019,
WEST BENGAL, INDIA.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc. **86**, 445-452 (1994)

ON FUZZY PRIMARY SUBMODULES

S. K. BHAMBRI, RAJESH KUMAR AND PRATIBHA KUMAR

(*Received 30 June 1993*)

Abstract : Fuzzy primary submodules are defined and studied in terms of their level submodules. Some properties of quotient of a fuzzy primary submodule and the intersection of two fuzzy primary submodules are discussed. Fuzzy cosets of fuzzy submodules are defined and the zero divisors in the module of all fuzzy cosets of a primary submodule are shown to be nilpotent.

Throughout the paper, R denotes a commutative ring with unity and M an R-module. Further it is assumed that the image set of every fuzzy submodule is finite.

1. Introduction. The concept of fuzzy submodules was first introduced by Negoita and Ralescu (1975) and subsequently studied, among others, by Fu Zheng Pan (1987).

In this paper, some properties of fuzzy primary submodules are studied. It turns out that a fuzzy submodule θ of M is fuzzy primary iff every level submodule of θ is primary except the whole module M (see Theorem 3.2). This result justifies the definition of the fuzzy primary submodule given in this paper. For a fuzzy submodule θ of M and an element $a \in M$, the fuzzy subset $(\theta : a)$ of R (see Definition 3.4) is shown to be a fuzzy ideal of R. In case, θ is fuzzy primary it turns out that $(\theta : a)$ is a fuzzy primary ideal of R (see Theorem 3.5). The radical of fuzzy ideal $(\theta : a)$ of R is also examined and under certain conditions, it is shown that $\sqrt{\theta : a} = \sqrt{\theta}$ [see Theorem 3.9]. A set of conditions under which the intersection of two fuzzy primary submodules is again a fuzzy primary submodule is also provided. Radical of intersection of two fuzzy primary submodules is also investigated.

Finally for a fuzzy primary submodule θ of M it is shown that every *zero divisor* in the module M_θ, consisting of all fuzzy cosets of θ in M, is nilpotent (see Theorem 4.4).

2. Preliminaries. In this section we give some basic definitions and results to be used later on.

DEFINITION 2.1. *Let μ be a fuzzy subset of a set S and let $t \in [0, 1]$. The set*

$$\mu_t = \{x \in S | \mu(x) \geq t\}$$

is called a level subset of μ.

DEFINITION 2.2. (Liu 1982). *A fuzzy subset μ of R is called a fuzzy ideal of R if*

(i) $\mu(x-y) \geq \min(\mu(x), \mu(y))$; *and*

(ii) $\mu(xy) \geq max(\mu(x), \mu(y))$ *for every* $x, y \in R$.

DEFINITION 2.3. (Kumar 1993) *A fuzzy ideal μ of R is called a fuzzy prime ideal of R if* $\mu(xy) \neq \mu(x)$ then $\mu(xy) = \mu(y), x, y \in R$.

DEFINITION 2.4. (Kumar 1993) *A fuzzy ideal μ of R is called fuzzy primary if for all x, y $\in R$, either* $\mu(xy) = \mu(x)$ *or* $\mu(xy) \leq \mu(y^n)$ for some $n \in Z_+$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEFINITION 2.5. (Kumar 1993) *Let μ be a fuzzy ideal of R. The fuzzy radical of μ, denoted by $\sqrt{\mu}$, is defined by*

$$\left(\sqrt{\mu}\right)(x) = \sup\left\{t \middle| x \in \sqrt{\mu_t}\right\};\ x \in R,\ t \in [0,1].$$

DEFINITION 2.6. (Bhambri, Kumar and Kumar 1993) *A fuzzy subset θ of M is called a fuzzy submodule of M if for all a, b $\in$ M and x $\in$ R,*

(i) $\theta(a+b) \geq min\{\theta(a), \theta(b)\}$;

(ii) $\theta(xa) \geq \theta(a)$.

For a fuzzy submodule θ of M, the level subsets $\theta_t = \{a \in M | \theta(a) \geq t\}$, $t \in Im\theta$, are submodules of M and are called level submodules of M.

DEFINITION 2.7. (Bhambri, Kumar, and Kumar 1993) *A fuzzy submodule θ of M is called a fuzzy prime submodule of M if $\theta(xa) = \theta(a)$; $0 \neq x \in R$ and $a \in M$.*

DEFINITION 2.8. (Atiyah and Macdonald 1969) *A submodule N of M is called primary if*

(i) *$N \neq M$ (ii) $xa \in N$, $a \notin N$, $x \in R$ implies that there exists $n \in Z_+$ such that $x^n M \subseteq N$.*

DEFINITION 2.9. (Atiyah and Macdonald 1969) *An element $x \in R$ is called a zero divisor in M if there exists $0 \neq a \in M$ such that $xa = 0$.*

DEFINITION 2.10. (Atiyah and Macdonald 1969) *An element $x \in R$ is called nilpotent element in M if there exists $n \in Z_+$ such that $x^n M = 0$.*

DEFINITION 2.11. (Atiyah and Macdonald 1969) *Let N be a submodule of M. The radical of N denoted by $\sqrt{N}$ is defined as :*

$$\sqrt{N} = \left\{x \in R \middle| x^q M \subseteq M \text{ for some } q \in Z_+\right\}.$$

DEFINITION 2.12. (Bhambri, Kumar and Kumar 1993). *Let θ be a fuzzy submodule of M. The radical of θ denoted by $\sqrt{\theta}$ is defined as :*

$$\sqrt{\theta}(x) = sup\left\{t \middle| x \in \sqrt{\theta_t}\right\}.$$

PROPOSITION 2.13. *Let θ be a fuzzy submodule of M. Then*

$$\sqrt{\theta_t} = \left(\sqrt{\theta}\right)_t,\ t \in \operatorname{Im}\theta.$$

Poof. Let $x \in \sqrt{\theta_t}$. Then $\sqrt{\theta}(x) \geq t$. So $x \in \left(\sqrt{\theta}\right)_t$. Therefore $\sqrt{\theta_t} \subseteq \left(\sqrt{\theta}\right)_t$. Conversely, let $x \in \left(\sqrt{\theta}\right)_t$. Let $\in > 0$ be given. Then there exists $i \in Z_+$ such that $t - \in < i$ and $x \in \sqrt{\theta_i}$. So $\theta_i \subseteq \theta_{t-\in}$. Therefore, $\sqrt{\theta_i} \subseteq \sqrt{\theta_{t-\in}}$. Thus $x \in \sqrt{\theta_{t-\in}}$ for all $\in > 0$. Hence $x \in \sqrt{\theta_t}$. Whence $\left(\sqrt{\theta}\right) \subseteq \sqrt{\theta_t}$. Conseqently $\sqrt{\theta_t} = \left(\sqrt{\theta}\right)_t$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

3. Fuzzy Primary Submodules

DEFINITION 3.1. *A non-constant fuzzy submodule θ of M is called a fuzzy primary submodule if $\theta(xa) > \theta(a)$, $a \in M$, $x \in R$, then $\theta(xa) \leq \theta(x^n b)$ for all $b \in M$ and for some $n \in Z_+$.*

It can be easily checked that a non-constant fuzzy prime submodule is always a fuzzy primary submodule.

THEOREM 3.2. *A fuzzy submodule θ of M is fuzzy primary iff θ_t is primary submodule of M for all $t \in Im\ \theta$ such that $\theta_t \neq M$.*

Proof. Let θ be a fuzzy primary submodule of M and let $t \in Im\ \theta$ such that $\theta_t \neq M$. Let $x \in R$, $a \in M$ such that $xa \in \theta_t$, $a \notin \theta_t$. Then $\theta(xa) \geq t$, $\theta(a) < t$. Clearly $\theta(xa) > \theta(a)$. Since θ is fuzzy primary, there exists $n \in Z_+$ such that $\theta(xa) \leq \theta(x^n b)$ for all $b \in M$. So $\theta(x^n b) \geq t$, for all $b \in M$ and for some $n \in Z_+$. Therefore, $x^n b \in \theta_t$ for all $b \in M$ and for some $n \in Z_+$. Hence $x^n M \subseteq \theta_t$. Whence θ_t is a primary submodule of M. Conversely let θ_t be primary submodule of M for all $t \in Im\ \theta$ such that $\theta_t \neq M$. Let $x \in R$, $a \in M$ be such that $t = \theta(xa) > \theta(a)$. Then $xa \in \theta_t$, $a \notin \theta_t$. Since θ_t is primary, there exists $n \in Z_+$ such that $x^n M \subseteq \theta_t$. That is $\theta(x^n b) \geq t = \theta(xa)$ for all $b \in M$. Hence θ is a fuzzy primary submodule of M.

COROLLARY 3.3. *Let N be a submodule of M, $N, \neq M$. Then the fuzzy sbmodule θ of M defined by*

$$\theta(x) = \begin{cases} 1, x \in N \\ \alpha, x \notin N, \ \alpha \in [0,1[\end{cases}$$

is fuzzy primary iff N is primary.

PROPOSITION 3.4. *Let θ be a fuzzy submodule of M and let $a \in M$.*

Then $(\theta : a)$ defined as :

$(\theta : a)(x) = \theta(xa)$, $x \in R$, is a fuzzy ideal of R.

Proof. Let $x, y \in R$. Then $(\theta : a)(x - y) = \theta((x - y)a)$

$$\geq min\ (\theta(xa), \theta(ya))$$

$$= min\ (\theta : a)(x), (\theta : a)(y))$$

Again, $(\theta{:}a)(xy) = \theta(xya) = \theta(x(ya)) \geq \theta(ya) = (\theta{:}a)(y)$

Similarly $(\theta{:}a)(xy) \geq \theta(xa) = (\theta{:}a)(x)$

So that, $(\theta{:}a)(xy) \geq \max((\theta{:}a)(x), (\theta{:}a)(y))$.

Hence $(\theta : a)$ is a fuzzy ideal of R.

The fuzzy ideal $(\theta : a)$ is termed as the *fuzzy quotient ideal* determined by θ and a.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

THEOREM 3.5. *Let θ be a fuzzy primary submodule of M and let $a \in M$. Then $(\theta : a)$ is a fuzzy primary ideal of R and $\sqrt{\theta}$ is a fuzzy prime ideal of R.*

Proof. Let $x, y \in R$ and let $(\theta : a)(xy) \neq (\theta : a)(x)$.

That is $\theta(xya) \neq \theta(xa)$. But $\theta(xya) = \theta(y(xa)) \geq \theta(xa) \geq \theta(a)$. Therefore $\theta(xya) > \theta(xa) \geq \theta(a)$. Since θ is fuzzy primary, there exists $n \in Z_+$ such that $\theta(y(xa)) \leq \theta(y^n b)$, for all $b \in M$. That is $(\theta:a)(xy) \leq \theta(y^n a) = (\theta:a)(y^n)$. Hence $(\theta : a)$ is a fuzzy primary ideal of R. Now let $\sqrt{\theta}(xy) \neq \sqrt{\theta}(y)$ and let $\sqrt{\theta}(xy) = t$. Then $xy \in \sqrt{\theta_t}$. This is $(xy)^n M = x^n y^n M \subseteq \theta_t$, for some $n \in Z_+$. Also $y^n M \not\subseteq \theta_t$, since $\sqrt{\theta}(y) \neq t$. Therefore there exists $a \in M$ such that $y^n a \notin \theta_t$ and $x^n y^n a \in \theta_t$. Since θ_t is a primary submodule of M, $(x^n)^m M \subseteq \theta_t$, for some $m \in Z_+$. So, $x \in \sqrt{\theta_t} = (\sqrt{\theta})_t$. Hence $s = (\sqrt{\theta})(x) \geq t$. But s cannot be greater than t, because otherwise xy would belong to $\sqrt{\theta_s}$ which is an obvious contradiction. Thus $s = \sqrt{\theta}(x) = \sqrt{\theta}(xy) = t$. consequently $\sqrt{\theta}$ is a fuzzy prime ideal of R.

PROPOSITION 3.6. *Let θ be a fuzzy submodule of M. then $\sqrt{\theta} \subseteq \sqrt{\theta:a}$ for all $a \in M$.*

Case (ii) $a \in \theta_t$. So $\theta(a) \geq t$. By hypothesis, $\theta(a) \leq \sqrt{\theta}(y)$ for all $y \in R$. Therefore $t \leq \sqrt{\theta}(x) = s$ (say). By Proposition 3.6, $t = \sqrt{(\theta:a)}(x) \geq \sqrt{\theta}(x) = s$. Hence $\sqrt{(\theta:a)}(x) = \sqrt{\theta}(x)$. Consequently $\sqrt{\theta:a} = \sqrt{\theta}$.

Proof. Straight forward.

In Results 3.10 and 3.11 we discuss the intersection of two fuzzy (fuzzy primary) submodules, respectively.

PROPOSITION 3.10. *Let θ and ϕ be fuzzy submodules of M and let $a \in M$, then $((\theta \cap \phi):a) = (\theta:a) \cap (\phi:a)$.*

THEOREM 3.11. *Let θ and ϕ be fuzzy primary submodules of M, and* let $\sqrt{(\theta:b)} = \sqrt{(\phi:b)}$, *for all $b \in M$. Then $\theta \cap \phi$ is a fuzzy primary submodule of M and $\sqrt{(\theta:b)} = \sqrt{(\phi:b)} = \sqrt{((\theta \cap \phi):b)}$. Further if $(\theta:b) \subseteq \sqrt{\theta}$ and $(\phi:b) \subseteq \sqrt{\phi}$, then $\sqrt{\theta \cap \phi} = \sqrt{\theta} = \sqrt{\phi}$.*

Proof. First we show that $\theta \cap \phi$ is a fuzzy primary submodule of M. Let $(\theta \cap \phi)(xa) > (\theta \cap \phi)(a)$, where $x \in R$, and $a \in M$. So $\min(\theta(xa), \phi(xa)) > \min(\theta(a), \phi(a)) = \theta(a)$ (say).

Therefore $\theta(xa) > \theta(a)$. Since θ is fuzzy primary, $\theta(xa) \leq \theta(x^n b)$, for all $b \in M$ and for some $n \in Z_+$. Hence $\min(\theta(xa)), \phi(xa) \leq \theta(x^n b) = (\theta:b)(x^n) \leq \sqrt{(\theta:b)}(x) = \sqrt{(\phi:b)}(x) = (\varphi:b)(x^m)$, (for some $m \in Z_+$), $= \phi(x^m b)$. Let $k = \max(m, n)$, so that $\theta(x^k b) = \theta(x^{k-n} x^n b) \geq \theta(x^n b) \geq \theta(xa)$ and $\phi(x^k b) = \phi(x^{k-m} x^m b) \geq \phi(x^m b) \geq \phi(xa)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Therefore $(\theta \cap \phi)(xa) = \min(\theta(xa), \phi(xa))$

$$\leq \min\left(\theta\left(x^k b\right), \phi\left(x^k b\right)\right)$$

$$= (\theta \cap \phi)\left(x^k b\right), \text{ for all } b \in M \text{ and } k \in Z_t.$$

Thus $\theta \cap \phi$ is a fuzzy primary submodule of M.

Now, let $\sqrt{((\theta \cap \phi):b)}(x) = t, x \in R$. Therefore $x^n \in \left((\theta \cap \phi):b\right)_t$, for some $n \in Z_+$. That is $((\theta \cap \phi):b)\left(x^n\right) = (\theta \cap \phi)\left(x^n b\right) \geq t$, and

Proof. Let $x \in R$ and let $\sqrt{\theta}(x) = s$. Then $x \in \sqrt{\theta_s}$. That is $x^n M \underline{C} \theta_s$ for some $n \in Z_+$. Therefore $x^n a \in \theta_s$, for all $a \in M$. So that $\theta\left(x^n a\right) \geq s$, which implies that $(\theta:a)\left(x^n\right) \geq s$. Hence $x \in \sqrt{(\theta:a)_s}$. Whence $\sqrt{\theta:a}(x) \geq s = \sqrt{\theta}(x)$. Thus $\sqrt{\theta}\,\underline{C}\sqrt{\theta:a}$.

REMARK 3.7. *We may point out that* $\sqrt{\theta} = \sqrt{\theta:a}$ *if* $(\theta:a)\underline{C}\sqrt{\theta}$.

In Theorem 3.9, we shall present another condition for the validity of the equality $\sqrt{\theta} = \sqrt{(\theta:a)}$.

PROPOSITION 3.8. *Let θ be a fuzzy submodule of M. Let $a \in M$ and let $\sqrt{(\theta:a)}(x) = t$ for some $x \in R$.* If $a \in \theta_t$, then $\sqrt{(\theta:a)}(x) = (\theta:a)(x) = \theta(a) = t$.

Proof. Since $\sqrt{(\theta:a)}(x) = t$, there exists $n \in Z_+$ such that $x^n \in (\theta:a)_t$. That is $(\theta:a)\left(x^n\right) = \theta\left(x^n a\right) \geq t$. Now $a \in \theta_t$ implies that $\theta(a) \geq t$, that is $(\theta:a)(1) \geq t$. So that $1 \in (\theta : a)_t$. Therefore $(\theta : a)_t = R$. Hence $t = \min\left\{(\theta:a)(y) | y \in R\right\}$.

Whence $\sqrt{(\theta:a)}(x) \leq (\theta:a)(y)$, for all $y \in R$.

In particular, $\sqrt{(\theta:a)}(x) \leq (\theta:a)(x) \leq \sqrt{(\theta:a)}(x)$.

Therefore $t = \sqrt{(\theta:a)}(x) = (\theta:a)(x) = \theta(xa) \geq \theta(a) \geq t$.

Hence $\sqrt{(\theta:a)}(x) = (\theta:a)(x) = \theta(a) = t$.

THEOREM 3.9. *Let θ be a fuzzy primary submodule of M. Let $a \in M$ be such that* $\theta(a) = \min\left\{\sqrt{\theta}(x) | x \in R\right\}$. *Then* $\sqrt{\theta:a} = \sqrt{\theta}$.

Proof. Let $x \in R$ and let $\sqrt{(\theta:a)}(x) = t$. Therefore $x \in \sqrt{(\theta:a)_t}$ and so $x^n \in (\theta:a)_t$ for some $n \in Z_t$. Hence $(\theta:a)\left(x^n\right) = \theta\left(x^n a\right) \geq t$. Therefore $x^n a \in \theta_t$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Case (i) $a \notin \theta_t$. As θ_t is fuzzy primary, there exists $m \in Z_+$ such that $(x^n)^m M \underline{C} \theta_t$. So $x \in \sqrt{\theta_t}$. Therefore, $\sqrt{\theta}(x) \geq t = \sqrt{(\theta:a)}(x)$. But $\sqrt{\theta}(x) \leq \sqrt{(\theta:a)}(x)$, (from Proposition 3.6.) Hence $\sqrt{\theta}(x) = \sqrt{(\theta:a)}(x)$,

so $\theta(x^n b) \geq t$ and $\phi(x^n b) \geq t$. Therefore $(\theta : b)(x^n) \geq t$ and $(\phi:b)(x^n) \geq t$. Hence $x^n \in (\theta:b)_t$ and $x^n \in (\phi:b)_t$. Whence $x \in \sqrt{(\theta:b)_t} \cap \sqrt{(\phi:b)_t}$. Let $\sqrt{(\theta:b)}(x) = s = \sqrt{(\phi:b)}(x)$. Since $x \in \sqrt{(\theta:b)_t}$, $s \geq t$. Let if possible $s > t$. Now $x \in \sqrt{(\theta:b)_s}$ implies $x^n \in (\theta:b)_s$. So that $(\theta:b)(x^n) = \theta(x^n b) \geq s$. Similarly $\phi(x^n b) \geq s$. Therefore $\min(\theta(x^n b), \phi(x^n b)) = (\theta \cap \phi)(x^n b)$ $= ((\theta \cap \phi):b)(x^n) \geq s$.

Which leads to $x \in \sqrt{((\theta \cap \phi):b)_s}$, a contradiction (since $s > t$).

Thus $s = t$ and we have $\sqrt{\theta:b} = \sqrt{\phi:b} = \sqrt{(\theta \cap \phi):b}$.

Further let $(\theta:b)\underline{C}\sqrt{\theta}$ and $(\phi:b)\underline{C}\sqrt{\phi}$, for all $b \in M$. Then by

Remark 3.7 and the hypothesis $\sqrt{\theta} = \sqrt{\phi}$. Now we show that $\sqrt{(\theta \cap \phi)} = \sqrt{\theta}$. Let $x \in R$ and let $\sqrt{(\theta \cap \phi)}(x) = t$, then $x \in \sqrt{(\theta \cap \phi)_t}$. Therefore $x \in \sqrt{\theta_t}$ and $x \in \sqrt{\phi_t}$.

Let $\sqrt{\theta}(x) = s = \sqrt{\phi}(x)$, then $s \geq t$, since $x \in \sqrt{\theta_t}$. Let if possible $s > t$.. Then as before $\min(\theta(x^n a), \phi(x^n a)) = (\theta \cap \phi)(x^n a) \geq s$, for all $a \in M$. Hence $x^n M \underline{C} (\theta \cap \phi)_s$, whence $x \in \sqrt{(\theta \cap \phi)_s}$, which is a contradiction. Thus $s = t$. Consequently $\sqrt{\theta} = \sqrt{\phi} = \sqrt{(\theta \cap \phi)}$.

4. Fuzzy Cosets

DEFINITION 4.1. *Let θ be a fuzzy submodule of M and let $m \in M$. Define* $\theta^*_m(m') = \theta(m' - m), m' \in M$.

Clearly θ^*_m is a fuzzy subset of M and is called the fuzzy coset determined by θ and m.

PROPOSITION 4.2. *The set $N = \{\theta^*_m | m \in M\}$ of all fuzzy cosets of a fuzzy submodule θ of M under the following compositions :*

$$\theta^*_m + \theta^*_n = \theta^*_{m+n}, m, n \in M,$$

$$x\theta^*_m = \theta^*_{xm}, x \in R,\ m \in M$$

is an R-module with $\theta^*_o = \theta$ as the additive identity and θ^*_{-m} as the additive inverse of θ^*_m

The proof of the above result can be easily obtained on the same lines as given in (Dixit, Kumar and Naseem 1992).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

LEMMA 4.3. *Let θ be a fuzzy submodule of M. Let $m, m' \in M$ and if $\theta(m) < \theta(m')$, then $\theta(m-m') = \theta(m) = \theta(m'-m)$.*

THEOREM 4.4. *Let θ be a fuzzy primary submodule of M. Let M_θ denote the set of all fuzzy cosets of θ in M. Then every zero divisor in M_θ is nilpotent.*

Proof. Let $x \in R$ be a zero divisor in M_θ, then there exists $\theta_m^* \in M_\theta$, $m \in M$ such that $x\theta_m^* = \theta_{xm}^* = \theta_o^* = \theta$, and $\theta_m^* \neq \theta_o^*$

That is $\theta_{xm}^*(0) = \theta(0) = t$ (say). then $\theta(xm) = t$. So $xm \in \theta_t$.

Now $\theta(m) \leq \theta(0) = t$. If $\theta(m) = t = \theta(0)$, then $\theta(m') \leq t$ for all $m' \in M$. Suppose $\theta(m') = t$ for some $m' \in M$.

Then $\theta(m'-m) \geq \min(\theta(m'), \theta(m)) = \theta(m') = t$.

So $\theta(m'-m) = t = \theta(m')$. If $\theta(m') < t = \theta(m)$, then by Lemma 4.3, $\theta(m'-m) = \theta(m')$. In any case $\theta(m'-m) = \theta(m')$, for all $m' \in M$.

Hence $\theta_m^*(m') = \theta(m'-m) = \theta(m') = \theta_o^*(m')$, for all $m' \in M$.

Whence $\theta_m^* = \theta_o^*$ a contradiction. Therefore $\theta(m) < t$.

That is $m \notin \theta_t$. Since θ_t is primary, $xm \in \theta_t$, $m \notin \theta_t$, there exists $n \in Z_+$ such that $x^n M \underline{C} \theta_t$. That is, $\theta(x^n a) \geq t$, for all $a \in M$. But $\theta(x^n a) \leq t$. So $\theta(x^n a) = t$ for all $a \in M$.

Therefore $\theta(m'-x^n a) = \theta(m')$, for all $m' \in M$, $a \in$ M. Now $\theta_x^* n_a(m') = \theta(m'-x^n a) = \theta(m')$ $= \theta_o^*(m')$, for all $m' \in M$. Therefore $\theta_x^* n_a = \theta_0^*$ for all $a \in M$. Thus $x^n \theta_a^* = \theta_o^*$ for all $a \in M$.

That is, $x^n \theta_M^* = \theta_o^*$ Hence x is nilpotent.

References

Atiyah M.F. and Macdonald I.G. (1969) : *"Introduction to Commutative Algebra"* Addison-Wesley Publishing Company, London.

Bhambri S.K., Kumar Rajesh and Kumar Pratibha, (1993) : "Fuzzy Prime Submodules and Radical of a Fuzzy Submodule" *Bull. Cal. Math. Soc. (to appear)*

Dixit V.N., Kumar Rajesh and Ajmal Naseem (1992) : "On Fuzzy rings", *Fuzzy Sets and Systems*, **49**, 205.

Kumar Rajesh, (1992) : "Certain fuzzy ideals of rings redefined", *Fuzzy Sets and Systems*, **46**, 251.

Kumar Rajesh, Bhambri S.K. and Kumar Pratibha, (1993) : "Fuzzy Submodules. Some Analogues and Deviations", *Fuzzy Sets and Systems (to appear)*

Kumar Rajesh (1993) : "On the fuzzy radical a fuzzy ideal", Fuzzy Sets and Systems, **55**, 233.

Liu Wang-jin (1982) : "Fuzzy invariant subgroups and fuzzy ideals", *Fuzzy Sets and Systems*, **8**, 133.

Negoita C.V. and Ralescu, D.A. (1975) : *Applications of Fuzzy Sets to System Analysis* Birkhauser, Basel.

Pan Fu-Zheng (1987) : "Fuzzy finitely generated modules" *Fuzzy Sets and Systems*, **21**, 105.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Pan Fu-Zheng (1988) : "Fuzzy Quotient Modules", *Fuzzy Sets and Systems*, **28**, 85.
Zadeh L.A. (1965) : "Fuzzy Sets" *Inform and Control*, **8**, 338.

DEPARTMENT OF MATHEMATICS,
K.M. COLLEGE (UNIVERSITY OF DELHI)
DELHI-110007, INDIA

DEPARTMENT OF MATHEMATICS
P.G.D.A.V. COLLEGE (E) (UNIVERSITY OF DELHI)
NEHRU NAGAR, RING ROAD
NEW DELHI-110065, INDIA

DEPARTMENT OF MATHEMATICS
K.M. COLLEGE (UNIVERSITY OF DELHI)
DELHI-110007, INDIA

MALIING ADDRESS :
(DR.) S.K. BHAMBRI
14 A/27 W.E.A.
KAROL BAGH
NEW DELHI-110005, INDIA

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc. **86**, 453-462 (1994)

THE BIORDER AND SANDWICH SETS ON A SEMIGROUP WITH IDEMPOTENTS

YU BINGJUN

(*Received 10 July 1993*)

Abstract : Two quasiorders ω^l and ω^r and on an arbitrary semigroup are defined. The sandwich sets on regular elements defined by Nambooripad and Pastijn are generalized to any elements of a semigroup which contains idempotents. The properties and structure of a sandwich set are characterized. As an application, three kinds of 'liftting properties' of a congruence on a semigroup with respect to regular elements are defined and investigated. Some important results about regular semigroups are generalized to wider classes of semigroups which contain idempotents.

1. Definition, Properties and Structure. In this paper, S always means a semigroup which contains idempotents, $E(S)$ the set of all idempotents of S, Reg S the set of all regular elements of S. Other terminologies and notations used in this paper, except specially announced, are all the same as those in (Clifford and Preston 1961), (Howie 1976) or (Nambooribad 1979).

DEFINITION 1. *Let S be an arbitrary semigroup. The following pair of binary relations on S is called the biorder on S :*

$$\omega^l = \left\{(a,b) \in S \times S : \exists s \in S^1, a = sb\right\},$$

$$\omega^r = \left\{(a,b) \in S \times S : \exists s \in S^1, a = bs\right\}$$

Moreover, we write $\omega = \omega^l \cap \omega^r$ *and*

$$\omega^l(a) = \left\{b \in S : b\omega^l a\right\},\ \omega^r(a) = \left\{b \in S : b\omega^r a\right\} \text{ and}$$

$$\omega(a) = \omega^l(a) \cap \omega^r(a),\ \forall a \in S.$$

The following statements are immidiate :

(1) ω^l *and* ω^r are both quasiorders on S.

(2) For all a, b, $s \in S$, we have

$$(a,b) \in \omega^l \Rightarrow (as, bs) \in \omega^l \text{ and } (a,b) \in \omega^r \Rightarrow (sa, sb) \in \omega^r$$

(3) *If* $\mathcal{L}, \mathcal{R}$ and $\mathcal{H}$ are the Green's relations on S, then

$$\omega^l \cap \left(\omega^l\right)^{-1} = \mathcal{L},\ \omega^r \cap \left(\omega^r\right)^{-1} = \mathcal{R},\ \omega \cap (\omega)^{-1} = \mathcal{H}.$$

(4) If $e \in E(S)$, then for all $a \in S$, $(a,e) \in \omega^l \Leftrightarrow a = ae$, $(a,e) \in \omega^r \Leftrightarrow a = ea$. In particular, for $f \in E(S), f\omega e \Leftrightarrow f \leq e$.

(5) If $e \in E(S)$, then for all $a \in S$,

$$(e,a) \in \omega^l \Rightarrow ae \in L_e \cap \omega(a), (e,a) \in \omega^r \Rightarrow ea \in R_e \cap \omega(a).$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEFINITION 2. *Let $n \geq 2$ $(a_1,\ldots,a_n)$ be an ordered sequence of elements of S. The following three sets*

$$\mathcal{M}(a_1,\ldots,a_n)=\{(g_1,\ldots,g_{n-1}):$$
$$g_i \in E(S)\cap\omega^l(a_i)\cap\omega^r(a_{i+1}),\ i=1,\ldots,n-1,$$
$$g_j a_{j+1}=a_{j+1}g_{j+1}, j=1,\ldots,n-2\},$$
$$S(a_1,\ldots,a_n)=\{(h_1,\ldots,h_{n-1})\in\mathcal{M}(a_1,\ldots,a_n):$$
$$\forall(g_1,\ldots,g_{n-1})\in\mathcal{M}(a_1,\ldots,a_n),$$
$$a_1g_1\omega^r a_1h_1, g_{n-1}a_n\omega^l h_{n-1}a_n\},$$
$$S_1(a_1,\ldots,a_n)=\{(h_1,\ldots,h_{n-1})\in\mathcal{M}(a_1,\ldots,a_n):$$
$$a_1h_1a_2h_2\ldots a_{n-1}h_{n-1}a_n=a_1a_2\ldots a_{n-1}a_n\}$$

are called the M-set, sandwich set and inner sandwich set of $(a_1,\ldots,a_n)$ in S, *respectively.*

The following proposition is easily verified :

PROPOSITION 1. We have the following :

(1) $(g_1,\ldots,g_{n-1})\in\mathcal{M}(a_1,\ldots,a_n)$ *if and only if*

$$g_i\in E(S)\cap\omega^l(a_1\ldots a_i)\cap\omega^r(a_{i+1}\ldots a_n), \forall i=1,\ldots,n-1, \text{ and}$$
$$g_j a_{j+1}=a_{j+1}g_{j+1}, \forall_j=1,\ldots n-2.$$

If $(h_1,\ldots,h_{n-1})\in\mathcal{M}(a_1,\ldots,a_n)$, *then*

(2) $a_1\ldots a_ih_ia_{i+1}\ldots a_n=a_1h_1a_2h_2\ldots a_{n-1}h_{n-1}a_n,\ \forall i=1,\ldots,n-1.$

(3) $a_1h_1\,\mathcal{R}\,a_1h_1a_2h_2\ldots h_{n-1}a_n\,\mathcal{L}\,h_{n-1}a_n.$

(4) *There exist* $a_1',\ldots,a_n'\in S$ *such that*

$$a_1'\in V(a_1h_1)\cap R_{h_1}, a_n'\in V(h_{n-1}a_n)\cap L_{h_{n-1}} \text{ and}$$
$$a_j'\in V(a_jh_j)\cap R_{h_j}\cap L_{h_{j-1}}, j=2,\ldots,n-1;$$

Furthermore, for any such n elements of S, we have

$$a_n'\ldots a_1'\in V(a_1h_1a_2h_2\ldots h_{n-1}a_n).$$ □

COROLLARY 2. (Pastijn 1980) (Proposition 3.3) *For any* $e_1,\ldots,e_n\in E(S), n\geq 2$, *if* $(h_1,\ldots,h_{n-1})\in\mathcal{M}(e_1,\ldots,e_n)$, *write* $a=e_1\ldots e_n, a'=h_{n-1}\ldots h_1$ *then we have*

$$a'aa'=a',\ a'\in V(e_1h_1e_2h_2\ldots h_{n-1}e_n) \text{ and}$$
$$aa'=e_1h_1,\qquad a'a=h_{n-1}e_n.$$

For any $a\in S$ write $U(a)=\{a'\in S: a'aa'=a'\}$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

THEOREM 3. *For any* $a_1,\dots,a_n \in S, n \geq 2$, *write* $a = a_1\dots a_n$, *then we have*

$$\mathcal{M}(a_1,\dots,a_n) = \{(h_1,\dots,h_{n-1}) : h_i = a_{i+1}\dots a_n a' a_1\dots a_i,$$
$$i = 1,\dots,n-1,\ a' \in U(a)\}.$$

Proof. Let $\overline{M}(a_1,\dots,a_n)$ denote the set of the right-hand of the equation in the theorem. It is easy to verify that $\overline{M}(a_1,\dots,a_n) \subseteq \mathcal{M}(a_1,\dots,a_n)$. Now let $(h_1,\dots,h_{n-1}) \in \mathcal{M}(a_1,\dots,a_n), b = a_1h_1a_2h_2\dots h_{n-1}a_n$. Since

$$h_1 \in E(S) \cap \omega^l(a_1) \cap \omega^r(a_2\dots a_n),$$

$$h_{n-1} \in E(S) \cap \omega^l(a_1\dots a_{n-1}) \cap \omega^r(a_n),$$

there exist $t_1, t_{n-1} \in S^1$, such that

$$h_1 = a_2\dots a_n t_1 a_1,\ h_{n-1} = a_n t_{n-1} a_1\dots a_{n-1}.$$

Let $b_1 = t_1 b t_1$, $b_2 = t_{n-1} b t_{n-1}$, It is easily verified that $b_j a b_j = b_j$, $ab_j a = b, j = 1, 2$; and $ab_1a_1 = a_1h_1$, $a_n b_2 a = h_{n-1}a_n$. Write $a' = b_2 a b_1$, we have $a' \in U(a) \cap V(b)$. So, if we write $g_i = a_{i+1}\dots a_n a' a_1\dots a_i$, $i = 1,\dots, n-1$, it suffices to prove $h_i = g_i$, $\forall i = 1,\dots,n-1$.

First, let $h_0 = ba'$, $h_n = a'b$, it is easily seen that

$$h_0\omega^r a_1,\ h_n\omega^l a_n, \text{ and}$$

$$a' \in V(b) \cap L_{h_0} \cap R_{h_n}.$$

One easily verifies that $h_{j-1}\mathcal{R}h_{j-1}a_j\mathcal{L}h_j$ so that there exist $a'_j \in V(a_jh_j) \cap L_{h_{j-1}} \cap R_{h_j}$, $\forall_j = 1,\dots,n$. By Proposition 1(4) we have

$$a'_n\dots a'_1 \in V(b) \cap L_{a'_1} \cap R_{a'_n} = V(b) \cap L_{h_0} \cap R_{h_n}.$$

Hence $a' = a'_n\dots a'_1$.

Secondly, we prove by induction that

$h_i = a_{i+1}\dots a_n a'_n\dots a'_{i+1} = a'_i\dots a'_1 a_1\dots a_i$, $\forall i = 1,\dots,n-1$. Indeed, since

$$a'_1 \in V(a_1h_1) \cap L_{h_0} \cap R_{h_1} \text{ and } h_0a_1 = a_1h_1,$$

we have $a'_1a_1 = a'_1h_0a_1 = a'_1a_1h_1 = h_1$; Dually, $a_na'_n = h_{n-1}$. Let $h_{i-1} = a'_{i-1}\dots a'_1a_1\dots a_{i-1}$, $h_{i+1} = a_{i+2}\dots a_na'_n\dots a'_{i+2}$. From $a'_i \in V(a_ih_i) \cap R_{h_i}$ and $a'_{i+1} \in V(a_{i+1}h_{i+1}) \cap L_{h_i}$ we have

$$h_i = a'_ia_ih_i = a'_ih_{i-1}a_i = a'_ia'_{i-1}\dots a'_1a_1\dots a_{i-1}a_i;$$

$$h_i = a_{i+1}h_{i+1}a'_{i+1} = a_{i+1}a_{i+2}\dots a_na'_n\dots a'_{i+2}a'_{i+1}.$$

Hence we have

$$g_i = a_{i+1}\dots a_n a' a_1\dots a_i$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$= (a_{i+1}\dots a_n a'_n \dots a'_{i+1})(a'_i \dots a'_1 a_1 \dots a_i)$$

$$= h_i^2 = h_i.$$

This completes the proof. □

THEOREM 4. *For any* $a_1,\dots,a_n \in S$, $n \geq 2$, *write* $a = a_1 \dots a_n$, *then*

$$S_1(a_1,\dots,a_n) = \{(h_1,\dots,h_{n-1}) : h_i = a_{i+1}\dots a_n a' a_1 \dots a_i,$$
$$a' \in V(a)\ i = 1,\dots,n-1\}.$$

Proof. By $S_1(a_1,\dots,a_n)$ we denote the set of the right-hand of the equation of the theorem. It is easy to verify that $\overline{S}_1(a_1,\dots,a_n) \subseteq S_1(a_1,\dots,a_n)$. Now let $(h_1,\dots,h_{n-1}) \in S_1(a_1,\dots,a_n)$, then $b = a_1 h_1 a_2 h_2 \dots h_{n-1} a_n = a_1 \dots a_n = a$. From the proof of Theorem 3, there is $a' \in V(b) = V(a)$ such that $h_i = a_{i+1}\dots a_n a' a_1 \dots a_i$, $i = 1,\dots,n-1$. This completes the proof. □

COROLLARY 5. *We have the following*

(1) *For any* $a_1,\dots,a_n \in S, n \geq 2$, $a_1 \dots a_n \in Reg\ S \Leftrightarrow S_1(a_1,\dots,a_n) \neq \phi$.

(2) *Denote* $E = E(S)$, *then* $V(E^n \cap Reg\ S) = E^{n+1} \cap Reg\ S$.

Proof. The statement (1) is clear, so we prove only the statement (2). Assume $e_1 \dots e_{n+1} \in E^{n+1} \cap Reg\ S$. By (1) there is

$$(h_1,\dots,h_n) \in S_1(e_1,\dots,e_{n+1}) \subseteq \mathcal{M}(e_1,\dots,e_{n+1}).$$

From Corollary 2 we have $e_1 \dots e_{n+1} = e_1 h_1 e_2 h_2 \dots h_n e_{n+1} \in V(h_n \dots h_1)$. So $E^{n+1} \cap Reg\ S \subseteq V(E^n \cap Reg\ S)$.

Conversely, let $v \in V(e_1 \dots e_n)$, $e_i \in E$, $i = 1,\dots,n$. Then there is $(h_1,\dots,h_{n-1}) \in S_1(e_1,\dots,e_n)$ by (1). Let $h_0 = e_1 \dots e_n v, h_n = v e_1 \dots e_n$. Since $h_{n-1} \dots h_1 \in V(e_1 \dots e_n)$ from Corollary 2, we have

$$v = v e_1 \dots e_n v = v e_1 \dots e_n h_{n-1} \dots h_1 e_1 \dots e_n v$$
$$= h_n h_{n-1} \dots h_1 h_0.$$

So $V(E^n \cap Reg\ S) \subseteq E^{n+1} \cap Reg\ S$. □

To describe the structure of a sandwich set $S(a_1,\dots,a_n)$, we denote the set of the i-th components of all elements in it by $S^{(i)}(a_1,\dots,a_n)$. We have the following :

PROPOSITION 6. *If* $S(a_1,\dots,a_n) \neq \phi$, $S^{(i)}(a_1,\dots,a_n)$ *is a rectangular subband of* S *for each* $i = 1,\dots,n-1$.

Proof. Let $(h_1,\dots,h_{n-1}), (g_1,\dots,g_{n-1}) \in S(a_1,\dots,a_n)$. By the definition we have $a_1 h_1 \mathcal{R} a_1 g_1$, $h_{n-1} a_n \mathcal{L} g_{n-1} a_n$. From this and Proposition 1(3), we have

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$b_1 = a_1h_1a_2h_2\ldots h_{n-1}a_n \mathcal{H}\, a_1g_1a_2g_2\ldots g_{n-1}a_n = b_2$$

Write $a = a_1\ldots a_n$, by Theorem 3, there are $a', a'' \in U(a)$ such that $h_i = a_{i+1}\ldots a_n a' a_1\ldots a_i$, $g_i = a_{i+1}\ldots a_n a'' a_1\ldots a_i$, $i = 1,\ldots,n-1$. Obviously $aa'a = b_1\mathcal{H}b_2 = aa''a$, so $aa' \mathcal{R}\, aa'', a'a \mathcal{L}\, a''a$. Let $\ddot{a} = a'aa''$, $e_i = a_{i+1}\ldots a_n\ddot{a}a_1\ldots a_i$, $i = 1,\ldots n-1$. One verifies that $\ddot{a} \in U(a)$, thus $(e_1,\ldots,e_{n-1}) \in \mathcal{M}(a_1,\ldots,a_n)$. Since

$$a_1e_1 = a\ddot{a}a_1 = aa'aa''a_1 = aa''a_1 = a_1g_1,$$
$$e_{n-1}a_n = a_n\ddot{a}a = a_na'aa''a = a_na'a = h_{n-1}a_n,$$

we have $(e_1,\ldots,e_{n-1}) \in S(a_1,\ldots,a_n)$.

Further, we have $h_ig_i = a_{i+1}\ldots a_na'aa''a_1\ldots a_i = e_i \in S^{(i)}(a_1,\ldots,a_n)$ and

$$h_ig_ih_i = e_ih_i = a_{i+1}\ldots = a_na'aa''aa'a_1\ldots a_i$$
$$= a_{i+1}\ldots a_na'a_1\ldots a_i = h_i$$

This completes the proof. □

THEOREM 7. *For any $a_1,\ldots,a_n \in S$, $n \geq 2$, we have $S_1(a_1,\ldots,a_n) \subseteq S(a_1,\ldots,a_n)$, and if $S_1(a_1,\ldots,a_n) \neq \phi$, then $S_1(a_1,\ldots,a_n) = S(a_1,\ldots,a_n)$.*

Proof. Let $(h_1,\ldots,h_{n-1}) \in S_1(a_1,\ldots,a_n)$, we have

$$a_1\ldots a_n = a_1h_1a_2\ldots a_n = a_1\ldots a_{n-1}h_{n-1}a_n.$$

For any $(g_1,\ldots,g_{n-1}) \in \mathcal{M}(a_1,\ldots,a_n)$, by $g_1\omega^r a_2\ldots a_n$ and the left compatibility of ω^r with respect to the operation on S, we have

$$a_1g_1\omega^r a_1a_2\ldots a_n = a_1h_1a_2\ldots a_n\omega^r a_1h_1.$$

Dually, $g_{n-1}a_n\omega^l h_{n-1}a_n$. Hence $(h_1,\ldots,h_{n-1}) \in S(a_1,\ldots,a_n)$.

Now, let $(\overline{h}_1,\ldots,\overline{h}_{n-1}) \in S(a_1,\ldots,a_n)$. Since

$$\overline{h}_1 \in \omega^l(a_1) \cap \omega^r(a_2\ldots a_n),$$

there are $t, s \in S^1$ such that $\overline{h}_1 = ta_1 = a_2\ldots a_ns$. We have

$$\begin{aligned} a_1\overline{h}_1a_2\overline{h}_2\ldots\overline{h}_{n-1}a_n &= a_1\overline{h}_1a_2\ldots a_n \\ &= a_1a_2\ldots a_nsta_1a_2\ldots a_n \\ &= a_1h_1a_2\ldots a_nsta_1h_1a_2\ldots a_n \\ &= a_1h_1\overline{h}_1h_1a_2\ldots a_n \\ &= a_1h_1a_2\ldots a_n \qquad \text{(By Proposition 6)} \\ &= a_1\ldots a_n. \end{aligned}$$

Thus $(\overline{h}_1,\ldots,\overline{h}_{n-1}) \in S_1(a_1,\ldots,a_n)$. □

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

There is an example that $S_1(a_1,\ldots,a_n)=\phi$, but $S(a_1,\ldots,a_n)\neq\phi$: Let $X=\{a,b\}$, $S=(X^+)^0$. Obviously, for any non-zero elements $a_1,\ldots,a_n\in S, n\geq 2$, $S(a_1,\ldots,a_n)=\{(0,\ldots 0)\}\neq\phi$, however, $S_1(a_1,\ldots,a_n)=\phi$, for the product $a_1\ldots a_n$ is not regular.

COROLLARY 8. *If $S_1(a_1,\ldots,a_n)\neq\phi$, the set $S_1^{(i)}(a_1,\ldots,a_n)$ is a rectangular sub-band of S, $i=1,\ldots,n-1$. Moreover,*

$$a_1h_1^{(1)}a_2h_2^{(2)}\ldots a_{n-1}h_{n-1}^{(n-1)}a_n=a_1\ldots a_n$$

for any $n-1$ elements $\left(h_1^{(k)},\ldots,h_{n-1}^{(k)}\right)\in S_1(a_1,\ldots,a_n), k=1,\ldots,n-1.$ □

Let

$$RM(a_1,\ldots,a_n)=\{x:x=a_1h_1a_2h_2\ldots h_{n-1}a_n,(h_1,\ldots,h_{n-1})\in\mathcal{M}(a_1,\ldots,a_n)\};$$

$$RS(a_1,\ldots,a_n)=\{x:x=a_1h_1a_2h_2\ldots h_{n-1}a_n,(h_1,\ldots,h_{n-1})\in S(a_1,\ldots,a_n)\},$$

and $RW(x)=\omega(x)\cap RegS$. Then we have the following :

PROPOSITION 9. *For any $a_1,\ldots,a_n\in S, n\geq 2$, write $a=a_1\ldots a_n$, then*

(1) $RM(a_1,\ldots,a_n)\subseteq RW(a)$.

(2) $RS(a_1,\ldots,a_n)\subseteq H(a_1,\ldots,a_n)$, *where $H(a_1,\ldots,a_n)$ is an $\mathcal{H}$-class of S in $RW(a)$.*

(3) *We have*

$$S(a_1,\ldots,a_n)=\{(h_1,\ldots,h_{n-1})\in\mathcal{M}(a_1,\ldots,a_n):$$
$$a_1h_1a_2h_2\ldots h_{n-1}a_n\in H(a_1,\ldots,a_n)\}.$$

(4) $RW(b)=RW(a)$, *for all $b\in RS(a_1,\ldots,a_n)$.*

Proof. The statements (1) and (2) are consequences of Proposition 1. To prove (3), let $(h_1,\ldots,h_{n-1})\in S(a_1,\ldots,a_n)$, $b=a_1h_1a_2h_2\ldots h_{n-1}a_n$. We have $b\in RW(a)$ from (1), so $RW(b)\subseteq RW(a)$ by the transitivity of ω. Now let $c\in RW(a)$. Since $c\omega^l a$, there is $a_0\in S$ such that $c=a_0a=a_0a_1\ldots a_n$. Since $c\in Reg\,S$, there is $(\overline{h}_0,\overline{h}_1,\ldots,\overline{h}_{n-1})\in S_1(a_0,a_1,\ldots,a_n)$ such that $c=a_0\overline{h}_0a_1\overline{h}_1\ldots\overline{h}_{n-1}a_n$. Clearly $(\overline{h}_1,\ldots,\overline{h}_{n-1})\in\mathcal{M}(a_1,\ldots,a_n)$, thus we have

$$c\omega^l h_{n-1}a_n\omega^l h_{n-1}a_n\mathcal{L}b,$$

i.e. $c\in\omega^l(b)$. Similarly, we have $c\in\omega^r(b)$. Hence $c\in\omega(b)\cap RegS=RW(b)$. This completes the proof. □

2 Three Lifting Properties of Congruences. In this section, for convenience, we write $(a,b)^n$ for a consequence of the form $(a,b,a,b,\ldots a,b)$ in which there are n pairs of (a,b).

DEFINITION 3. *Let ρ be a congruence on S. ρ is said to be idempotent- [res. regular element-, mutually inverse elements-] lifting, simplified as l- [res. R-, MV-] lifting, if, for any e*

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$\in E(S/\rho)$ [res. $a' \in Reg\ (S/\rho)$, $a'b' \in S/\rho$, $b' \in V(a')$], there exists [res. exists, exist] $e \in E(S)$ [res. $a \in Reg\ (S/\rho)$, $a, b \in S$, $b \in V(a)$] such that $e\rho = e'$ [res. $a\rho = a'$, $a\rho = a'$, $b\rho = b'$]. A semigroup S is said to be I-[res. R-, MV-] lifting, if any congruence on it is I-[res. R-, MV-] lifting.

An element $a \in S$ is called quasi-regular, if there is an positive integer n such that $a^n \in Reg S$. S is called quasi-regular, if all of its elements are quasi-regular. From (Edwards 1983), any quasi-regular semigroup possesses the three lifting properties above. Here, we prove that each of these properties is exactly equivalent to a certain weaker quasi-regularity of elements of a semigroup.

THEOREM 10. *A congruence ρ on a semigroup S is I-lifting if and only if for any $e' \in E(S/\rho)$, there exist $a, b \in S$ such that $a\rho = b\rho = e'$ and the product ab is a quasi-regular element of S. More precisely, if $(ab)^n \in RegS$, $n \geq 1$, then $S^{(i)}(a,b)^n \subseteq e'\rho^{-1}, \forall i = 1,\ldots,2n-1$.*

Proof. The 'only if' part is obvious. Now let $a, b \in S$, $a\rho = b\rho = e'$ and $(ab)^n \in Reg\ S$, for some $n \geq 1$. From Corollary 5 and Theorem 7, $S\ (a, b)^n = S_1(a, b)^n \neq \phi$. For any $i = 1,\ldots,2n-1$, let $h \in S^{(i)}(a,b)^n$, If $i = 2k$, $k = 1,\ldots,n-1$, by Theorem 4 there is $v \in V\ ((ab)^n)$ such that $h = (ab)^{n-k} v\ (ab)^k$. Thus we have

$$\begin{aligned} h\rho &= (ab)^{n-k}\rho(v\rho)(ab)^k\rho = e'(v\rho)e' \\ &= (ab)^n\rho(v\rho)(ab)^n\rho = \left[(ab)^n v(ab)^n\right]\rho \\ &= (ab)^n\rho = (e')^n = e' \end{aligned}$$

If $i = 2k+1$, $k = 0, \ldots, n-1$, there is $v \in V\left((ab)^n\right)$ such that $h = b(ab)^{n-k-1}v(ab)^k a$. Similarly one can show that $h\rho = e'$. □

THEOREM 11. *Let ρ be a congruence on S. ρ is R-lifting if and only if for any $a' \in Reg\ (S/\rho)$, there exist $b' \in V(a')$ and $x, y \in S$ such that $x\rho = a'$, $y\rho = b'$ and $(xy)^2$ is a quasi-regular element in S. Furthermore, if this is the case, there is $x' \in RegS$ such that $x'\rho = a'$ and $x\omega x$.*

Proof. It suffices to prove 'if' part. Let $a' \in Reg(S/\rho), b' \in V(a'), x, y \in S$ such that $x\rho = a'$, $y\rho = b'$ and $(xy)^{2n} \in Reg\ S$. By the same arguments as the proof of Theorem 10 there exists $h \in S^{(2i)}(x,y)^{2n}$, $i = 1,\ldots,2n-1$. One can easily verify that $x' = hx \in Reg\ S$, $x'\rho = a'$ and $x'\omega x$. □

THEOREM 12. *Let ρ be a congruence on S. ρ is MV-lifting if and only if for any $a'b' \in S/\rho$ with $b' \in V(a')$, there exist $x, y \in S$ such that $x\rho = a'$, $y\rho = b'$ and $(xy)^2$ is a quasi-regular element in S. Furthermore, if this is the case, there exist x', $y' \in S$ such that $y' \in V(x')$, $x'\rho = a'$, $y'\rho = b'$, $x'\omega x$ and $y'\omega y$.*

Proof. It is enough to prove "if" part. Let a', b', x and y satisfy the conditions in the theorem. There is $n \geq 1$ such that $(xy)^{2n} \in Reg\ S$, so that $S_1(x,y)^{2n} \neq \phi$. Let

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$h \in S_1^{(2i)}(x,y)^{2n}, i = 1,\ldots,2n-1$. Then $h = (xy)^{2n-i} v (xy)^i$, for some $v \in V(xy)^{2n}$). Write $x' = hx$ and $y' = y(xy)^{2n-i-1} v (xy)^i$, one easily verifies that $y' \in V(x'), x'\rho = a', y'\rho = b', x'\omega x$ and $y'\omega y$.

It is evident that the MV-lifting property implies both I-and R-lifting properties. The following two examples show that neither the class of I-nor R-lifting congruences contains another one so that the class of MV-lifting congruences is a proper subclass of them.

Example 1. Let $S = \langle a \rangle^1$, where $\langle a \rangle$ is the infinite cyclic semigroup generated by a. Let $T = \langle b : b^3 = b \rangle$, and $\phi : S \to T$, $\phi(a^i) = b^i$, $\phi(1) = b^2$. ϕ is obviously a homomorphism from S onto T.Write $\rho = Ker\phi$. One easily verifies that ϕ is I-liftting but not R-lifting.

Example 2. Let $S = \langle p,q : pqp = p, qpq = q \rangle$, Let ρ be an equivalence relation on S whose equivalence-classes are pS^1p, pS^1q, qS^1p and qS^1q. One easily verifies that ρ is R-lifing but not I-lifting.

Following (Mitsch 1990), a semigroup S is called E-inversive, if, for any element $a \in S$, there is $x \in S$ such that $ax \in E(S)$. It is easy to verify that each of the three lifting classes of semigroups is a proper subclass of the class of E-inversive semigroups. The author, however, has not known if the class of quasi-regular semigroups is a proper subclass of the class MV-lifting ones or not. The author has not been aware of whether the three lifting classes are really different from each other.

In the remainder of this paper, we give another two applications to show that the concepts introduced in §1 are useful for some cases where the old ones are invalid.

The first is a generalization of Theorem 15 in (Hall 1972). For this we give a new proof of a lemma due to T.E., Hall which is much more concise than the original (due to W.D. Munn).

LEMMA 13. *Let ρ be a congruence on a semigroup S which is contained in $\mathcal{L}$. For any $(a, b) \in \rho$, H_a is a group if and only if H_b is a group.*

Proof. Assume that H_a is a group and a^{-1} is the inverse of a in H_a. Since $(aa^{-1}, ba^{-1}) \in \rho$, we have $\left(aa^{-1}, (ba^{-1})^2\right) \in \rho \subseteq \mathcal{L}$. This shows $(ba^{-1})^2 \in RegS$. By Theorem 10 there is $e \in S_1(ba^{-1}, ba^{-1}) \subseteq (aa^{-1})\rho o \rho^{-1}$, hence $e\mathcal{L}\, aa^{-1}\mathcal{L} a\mathcal{L} b$. Further, by theorem 4 there is $v \in V\left((ba^{-1})^2\right)$ such that $e = ba^{-1}vba^{-1}$. From this we have $b = be\, b^2 a^{-1} vba^{-1}$ so that $b\mathcal{R}b^2$. On the other hand, $(b^2, a^2) \in \rho \subseteq \mathcal{L}$, thus we have $b\mathcal{L}\, a\mathcal{L}\, a^2\mathcal{L}\, b^2$. So $b\mathcal{H}\, b^2$ which implies that H_b is a group. □

PROPOSITION 14. *Let S be a MV-lifting semigroup, ρ, a congruence on S contained in $\mathcal{L}$. For any $a \in S$, a is in a subgroup of S if and only if $a\rho$ is in a subgroup of S/ρ.*

Proof. It is sufficient to prove the "if" part. Assume that $a\rho$ is in a subgroup of S/ρ. There exists $(a\rho)^* \in V(a\rho)$ such that $a\rho(a\rho)^* = (a\rho)^* a\rho$. Thanks to the MV-lifting property there exist $x, x' \in S, x' \in V(x)$ such that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$x\rho = a\rho$, $x'\rho = (a\rho)^*$ and, $(xx')\rho = a\rho(a\rho)^* = (a\rho)^* a\rho = (x'x)\rho$

From this we get $(xx', x'x) \in \rho \subseteq \mathcal{L}$. Hence H_x mst be a group. But $(x,a) \in \rho, H_a$ is a group by Lemma 13. □

The second application is the following concept.

DEFINITION 4. *A congruence ρ on a semigroup S is called strongly MV-lifting, if, for any $a \in Reg\ S$ and any $y\rho \in V(a\rho)$ there exists $a' \in V(a)$ such that $b\rho = y\rho$. S is called strongly MV-lifting, if every congruence on it is strongly MV-lifting.*

The following proposition is a generalization of a corresponding result for regular semigroups, due to Nambooripad and Pastijn (1981).

PROPOSITION 15. *Let S be a quasi-regular semigroup. If the conditions*

$$\omega\mathcal{L} = \mathcal{L}\omega \qquad \text{and} \qquad \omega\mathcal{R} = \mathcal{R}\omega$$

hold on the biordered set E(S), then S is strongly MV-lifting.

Proof. Let ρ be a congruence on S, $a \in Reg\ S$ and $y\rho \in V(a\rho)$, for some $y \in S$. Take $f\rho \in E(R_{a\rho})$, where we can assume $f \in E(S)$ because of the I-liftting property of S. Let $g \in E(R_a)$, so that $g\rho\mathcal{R}f\rho$. Since S is quasi-regular, there is an integer n such that $(fg)^n \in RegS$. Thus there exists $h \in S_1\left((fg)^{n-1}f, g\right)$ One verifies that

$$h\rho \in S_1\left((fg)^{n-1}f\right)\rho, g\rho\Big) = S_1(f\rho, g\rho) = \{f\rho\}$$

Moreover, since $(h,g) \in \omega^r = \mathcal{R}\omega = \omega\mathcal{R}$, there is $e \in E(S)$ such that $h\omega e\mathcal{R}g\mathcal{R}a$. From this we have $e\rho = f\rho$. This shows that, for any $f\rho \in E\left(R_{a\rho}\right)$, there exists $e \in E(R_a)$ such that $e\rho = f\rho$. Dually, for any $f'\rho \in E(L_{a\rho})$, there exists $e' \in E(S)$ such that $e'\rho = f'\rho$.

Now, write $f\rho = a\rho y\rho$, $f'\rho = y\rho a\rho$, $f, f' \in E(S)$. From the preceding results, there exist $e \in E(R_a)$, $e' \in E(L_a)$ such that $e\rho = f\rho$, $e'\rho = f'\rho$. It is well-known that there is $a' \in V(a) \cap L_e \cap R_{e'}$. One easily verifies that

$$a'\rho \in V(a\rho),\ \ y\rho\mathcal{L}f\rho = e\rho\,\mathcal{L}\,a'\rho,$$
$$y\rho\,\mathcal{R}\,f'\rho = e'\rho\,\mathcal{R}\,a'\rho.$$

Hence we have $y\rho\,\mathcal{H}\,a'\rho$, which implies $a'\rho = y\rho$. This completes the proof. □

References

Clifford A.H. and Preston, G.B. (1961) : The Algebraic Theory of Semigroups, *Math. Srveys No.7., Amer. Math. Soc., Providence, R.I.*, I, II (1967).

Edwards, P.M. (1983) : Eventually regular semigroups, *Bull. Austral. Math. Soc.* **28**, 23.

Fitz-Gerald, D.G. (1972) : On inverses of products of idempotents in regular semigroups, *J. Austral. Math. Soc.* **13**, 335.

Hall, T.E. (1972) : Congruences and Green's relations on reglar semigroups, *Glasgow Math. J.* **13**.

Howie, J.M. (1976) : *An introduction to semigroup theory*, Academic Press, London.

Mitsch, H. (1990) : Subdirect products of E-inversive semigroups, *J. Austral. Math. Soc.* (Series A) **48**, 66.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Nambooripad, K.S.S. (1979) : Structure of regular semigroups I, *Mem. Amer. Math. Soc.*, 224.
Nambooripad, K.S.S. & Pastijn, F. (1981) : V-regular semigroups, *Proc. Roy. Soc. Edinburgh* **88a**, 257.
Pastijn, F. (1980) : The biorder on the groupoid of idempotents of a semigroup, *J. Algebra* **65**, 147.

DEPARTMENT OF APPLIED MATHEMATICS,
CHENGDU COLLEGE OF GEOLOGY
CHENGDU, SICHUAN-610059
PRC

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc. **86**, 463-468 (1994)

ON THE PRODUCT OF AN ARBITRARY FUNCTION AND THE FRACTIONAL DERIVATIVE OF DIRAC-DELTA DISTRIBUTION-II

GOPI AHUJA

(Received 30 July 1993; revision received 27 January 1994)

Abstract : In the present paper an expression for the product of an arbitrary function $f(x)$ and the fractional derivative of Dirac-delta distribution with respect to a continuously differentiable increasing or decreasing functional $\rho(x)$ is obtained. The result is illustrated by means of suitable examples.

1. Introduction. Riemann-Liouville and Weyl fractional integrals of order α are defined, for Re $\alpha > 0$ by

$$\left(I_x^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\, dt \qquad \text{..... (1.1)}$$

$$\left(K_x^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1} f(t)\, dt. \qquad \text{..... (1.2)}$$

These can be usefully generalized in two ways :

Firstly, through the operators of Kober (1940) and secondly we may wish to integrate them with respect to a continuously differentiable increasing or decreasing function ρ of a positive real variable x, yielding an expression of the type

$$\left(I_{\rho(x)}^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \left(\rho(x)-\rho(t)\right)^{\alpha-1} f(t)\rho'(t)\, dt \qquad \text{..... (1.3)}$$

$$\left(K_{\rho(x)}^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty \left(\rho(t)-\rho(x)\right)^{\alpha-1} f(t)\rho'(t)\, dt. \qquad \text{..... (1.4)}$$

For Re $\alpha < 0$, we defined $I_{\rho(x)}^\alpha$ as inverse of $I_{\rho(x)}^{-\alpha}$ and $K_{\rho(x)}^\alpha$ as the inverse of $K_{\rho(x)}^{-\alpha}$, Thus for negative α, $I_{\rho(x)}^\alpha$ is an operator of fractional differentiation of order α with respect to $\rho(x)$ i.e.

$$I_{\rho(x)}^{-\alpha} = D_{\rho(x)}^\alpha \qquad \text{..... (1.5)}$$

and

$$K_{\rho(x)}^{-\alpha} = (-1)^\alpha D_{\rho(x)}^\alpha ; D = \frac{d}{dx}. \qquad \text{..... (1.6)}$$

By making use of this definition of fractional derivative for $\rho(x) = x$, Grunwald (1867) and Letnikov (1868) found the Leibnitz rule for fractional derivative

$$D_x^\alpha (u.v) = \sum_{r=0}^\infty {}^\alpha C_r D_x^{\alpha-r}\, u\, D_x^r v, \qquad \text{..... (1.7)}$$

where

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$^{\alpha}C_r = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-r+1)\,\Gamma(r+1)}.$$

There is a disturbing feature of (1.7). It is obvious that

$$D_x^{\alpha}(u.v) = D_x^{\alpha}(v.u),$$

but this fact is not clear in the right hand side, since u is differentiated fractionally while v is differentiated in the usual elementary sense. To remove this anamoly, Osler (1970) has proved that

$$D_{\rho(x)}^{\alpha}\big(u(x)v(x)\big) = \sum_{r=-\infty}^{\infty} {}^{\alpha}C_{\gamma+r} D_{\rho(x)}^{\alpha-\gamma-r} u(x) D_{\rho(x)}^{\gamma+r} v(x) \qquad \text{..... (1.8)}$$

for all complex α and γ except negative integral α.

The above definition of fractional derivative given by (1.5) and (1.6) and the Leibnitz rule (1.8) suggests an improvement of certain results of one of my earlier paper (Ahuja 1992). In the present paper we obtain an expression for the product of an arbitrary function $f(x)$ and the fractional derivative of Dirac-delta distribution with respect to a continuously differentiable increasing or decreasing function $\rho(x)$. The results are illustrated by means of suitable examples.

§ (2). If ϕ is an element of a testing function space K (Gelfand and Shilov 1964,p.4), then it is easy to see that

$$< I_{\rho(x)}^{-\alpha} f, \phi > = < f, \rho'(x) K_{\rho(x)}^{-\alpha}\{\phi(x)/\rho'(x)\} > \qquad \text{..... (2.1)}$$

and

$$< K_{\rho(x)}^{-\alpha} f, \phi > = < f(x), \rho'(x) I_{\rho(x)}^{-\alpha}\{\phi(x)/\rho'(x)\} >. \qquad \text{..... (2.2)}$$

The αth derivative of Dirac-delta distribution $\delta(x)$ with respect to $\rho(x)$ can be defined as below :

For $\phi \in K$

$$\begin{aligned} < D_{\rho(x)}^{\alpha}\delta(x), \phi(x) > &= < I_{\rho(x)}^{-\alpha}\delta(x), \phi(x) > \\ &= < \delta(x), \rho'(x) K_{\rho(x)}^{-\alpha}\{\phi(x)/\rho'(x)\} > \\ &\qquad \text{(by (2.1))} \\ &= < \delta(x), (-1)^{\alpha}\rho'(x) D_{\rho(x)}^{\alpha}\{\phi(x)/\rho'(x)\} > \\ &\qquad \text{(by (1.6))} \\ &= (-1)^{\alpha}\rho'(x) D_{\rho(x)}^{\alpha}\{\phi(x)/\rho'(x)\}\Big|_{x=0}. \end{aligned} \qquad \text{..... (2.3)}$$

We are now in a position to obtain the required expression given in the form of the following theorem :

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

THEOREM (2.1) *If $\rho(x)$ is a continuously differentiable increasing or decreasing function and $f(x)$ is a function defined in the nbd. of the origin, whose derivatives of all order with respect to $\rho(x)$ are continuous at the origin, then the product $f(x)D^{\alpha}_{\rho(x)}\delta(x)$ exists*

and

$$f(x)D^{\alpha}_{\rho(x)}\delta(x) = \sum_{r=-\infty}^{\infty}(-1)^{2\alpha-\gamma-r}\,{}^{\alpha}C_{\gamma+r}D^{\gamma+r}_{\rho(x)}f(x)\Big|_{x=0}\; D^{\alpha-\gamma-r}_{\rho(x)}\delta(x).$$

Proof. We have for $\phi \in K$,

$$< f(x)D^{\alpha}_{\rho(x)}\delta(x), \phi(x) > \; =$$

$$=< D^{\alpha}_{\rho(x)}\delta(x), f(x)\phi(x) >$$

$$=< \delta(x), (-1)^{\alpha}\rho'(x)D^{\alpha}_{\rho(x)}\{f(x)\phi(x)/\rho'(x)\} >$$

(by (2.1))

$$=< \delta(x), (-1)^{\alpha}\rho'(x)\sum_{r=-\infty}^{\infty}{}^{\alpha}C_{\gamma+r}D^{\gamma+r}_{\rho(x)}f(x)D^{\alpha-\gamma-r}_{\rho(x)}\{\phi(x)/\rho'(x)\}$$

(by (1.8))

$$=(-1)^{\alpha}\sum_{r=-\infty}^{\infty}{}^{\alpha}C_{\gamma+r} < \delta(x), \rho'(x)D^{\gamma+r}_{\rho(x)}f(x)D^{\alpha-\gamma-r}_{\rho(x)}\{\phi(x)/\rho'(x)\} >$$

$$=(-1)^{\alpha}\sum_{r=-\infty}^{\infty}{}^{\alpha}C_{\gamma+r}\left\{\rho'(x)D^{\gamma+r}_{\rho(x)}f(x)\,D^{\alpha-\gamma-r}_{\rho(x)}\{\phi(x)/\rho'(x)\}\right\}\Big|_{x=0}$$

$$=(-1)^{\alpha}\sum_{r=-\infty}^{\infty}{}^{\alpha}C_{\gamma+r}D^{\gamma+r}_{\rho(x)}f(x)\Big|_{x=0} < \delta(x), \rho'(x)D^{\alpha-\gamma-r}_{\rho(x)}\{\phi(x)/\rho'(x)\} >$$

$$=(-1)^{\alpha}\sum_{r=-\infty}^{\infty}{}^{\alpha}C_{\gamma+r}D^{\gamma+r}_{\rho(x)}f(x)\Big|_{x=0} < K^{-(\alpha-\gamma-r)}_{\rho(x)}\delta(x), \phi(x) >$$

(by (2.2))

$$=< \sum_{r=-\infty}^{\infty}(-1)^{2\alpha-\gamma-r}\,{}^{\alpha}C_{\gamma+r}D^{\gamma+r}_{\rho(x)}f(x)\Big|_{x=0}\; D^{\alpha-\gamma-r}_{\rho(x)}\delta(x),\ \phi(x) >.$$

This implies that

$$f(x)D^{\alpha}_{\rho(x)}\delta(x) = \sum_{r=-\infty}^{\infty}(-1)^{2\alpha-\gamma-r}\,{}^{\alpha}C_{\gamma+r}D^{\gamma+r}_{\rho(x)}f(x)\Big|_{x=0}\; D^{\alpha-\gamma-r}_{\rho(x)}\delta(x), \qquad \text{..... (2.4)}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

which generalizes an expression obtained by the author (Ahuja 1992, p. 232) and also removes the anamoly, that one function is differentiated fractionally and the other is differentiated in usual way.

The expression (2.4) can also be written as

$$f(x)D^{\alpha}_{\rho(x)}\delta(x-\xi)=\sum_{r=-\infty}^{\infty}(-1)^{2\alpha-\gamma-r}\,{}^{\alpha}C_{\gamma+r}D^{\gamma+r}_{\rho(x)}f(x)\Big|_{x=\xi}\;D^{\alpha-\gamma-r}_{\rho(x)}\delta(x-\xi). \qquad \text{..... (2.5)}$$

In the light of the integral analogue of Leibnitz rule for fractional derivative (Osler 1972, p. 904), (2.4) and (2.5), we can write as

$$f(x)D^{\alpha}_{\rho(x)}\delta(x)=\int_{-\infty}^{\infty}(-1)^{2\alpha-\gamma-w}\,{}^{\alpha}C_{\gamma+w}D^{\gamma+w}_{\rho(x)}f(x)\Big|_{x=0}\;D^{\alpha-\gamma-w}_{\rho(x)}\delta(x)\,dw \qquad \text{..... (2.6)}$$

$$f(x)D^{\alpha}_{\rho(x)}\delta(x-\xi)=\int_{-\infty}^{\infty}(-1)^{2\alpha-\gamma-w}\,{}^{\alpha}C_{\gamma+w}D^{\gamma+w}_{\rho(x)}f(x)\Big|_{x=\xi}\;D^{\alpha-\gamma-w}_{\rho(x)}\delta(x-\xi)\,dw. \qquad \text{..... (2.7)}$$

Kanwal (1983, 5, p. 40.) has studied an expression for the product of $f(x)$ and the nth derivative of Dirac-delta distribution and applied his result to obtain the solution of the differential equation

$$<Lf,\phi>=<\sum_{|k|\le p}a_kD^kf,\phi> \;=\; <f,\sum_{|k|<p}(-1)^{|k|}D^k(a_k\phi)>$$

$$=<f,L^*\phi>. \qquad \text{..... (2.8)}$$

A natural question arises—can we solve differential equation of the type (2.8) for fractional orders? This problem is still under investigation. The expressions obtained here, gives the hope that one may be able to apply them to solve a differential equation of fractional order, also.

§ 3. In this section, the results obtained in § 2 are illustrated by means of suitable examples.

EXAMPLE (1) If $f(x)=e^x$

$\rho(x)=1-x$

then $D^{\gamma+r}_{1-x}e^x=(-1)^{\gamma+r}e^x$

and at $x=0$, $D^{\gamma+r}_{1-x}e^x=(-1)^{\gamma+r}$.

Thus from (2.4) we have for all complex α and γ, except for negative integral α,

$$e^xD^{\alpha}_{1-x}\delta(x)=\sum_{r=-\infty}^{\infty}(-1)^{2\alpha-\gamma-r}\,{}^{\alpha}C_{\gamma+r}(-1)^{\gamma+r}D^{\alpha-\gamma-r}_{1-x}\delta(x)$$

$$=(-1)^{2\alpha}\sum_{r=-\infty}^{\infty}{}^{\alpha}C_{\gamma+r}D^{\alpha-\gamma-r}_{1-x}\delta(x).$$

and from (2.6) we have

$$e^xD^{\alpha}_{1-x}\delta(x)=(-1)^{2\alpha}\int_{-\infty}^{\infty}{}^{\alpha}C_{\gamma+w}D^{\alpha-\gamma-w}_{1-x}\delta(x)dw.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

EXAMPLE (2). If $f(x)=\dfrac{\log x}{x}$

$$\rho(x)=x,\ \gamma=0.$$

then

$$D_x^r\left\{\frac{\log x}{x}\right\}=(-1)^r r!\,x^{-r-1}\left\{\log x-\sum_{m=1}^{r}\frac{1}{m}\right\}.$$

Thus from (2.5), we have for all complex α, except for negative integral α,

$$\left\{\frac{\log x}{x}\right\}D_x^\alpha\delta(x-\xi)=\sum_{r=-\infty}^{\infty}(-1)^{2\alpha}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-r+1)}\xi^{-r-1}\left\{\log\xi-\sum_{m=1}^{r}\frac{1}{m}\right\}\times D_x^{\alpha-r}\delta(x-\xi).$$

Its integral analogue, from (2.7) can be written as —

$$\left\{\frac{\log x}{x}\right\}D_x^\alpha\delta(x-\xi)=(-1)^{2\alpha}\Gamma(\alpha+1)\int_{-\infty}^{\infty}\frac{\xi^{-w-1}}{\Gamma(\alpha-w+1)}\left\{\log\xi-\int_1^w\frac{1}{t}dt\right\}\times D_x^{\alpha-w}\delta(x-\xi)dw$$

EXAMPLE (3). Let $f(x)=x^{\upsilon-1}e^{ax}$, Re $\upsilon>0$

$$\rho(x)=x,$$

then from (Erde'lyi 1955, p. 182)

$$D_x^{\gamma+r}\left\{x^{\nu-1}e^{ax}\right\}\Big|_{x=\xi}=\frac{\Gamma(\nu)}{\Gamma(\nu-\gamma-r)}\xi^{\nu-\gamma-r-1}{}_1F_1(\nu;\nu-\gamma-r;a.\xi).$$

Thus for all complex α and γ, except for negative integral α, (2.5) becomes—

$$\left\{x^{\nu-1}e^{ax}D_x^\alpha\delta(x-\xi)\right\}=\sum_{r=-\infty}^{\infty}(-1)^{2\alpha-\gamma-r}\frac{\Gamma(\nu)}{\Gamma(\nu-\gamma-r)}{}^{\alpha}C_{\gamma+r}\times\xi^{\nu-\gamma-r-1}{}_1F_1(\nu;\nu-\gamma-r;a\xi)D_x^{\alpha-\gamma-r}\delta(x-\xi)$$

and the integral analogue (2.7) becomes

$$\left\{x^{\nu-1}e^{ax}D_x^\alpha\delta(x-\xi)\right\}=(-1)^{2\alpha-\gamma}\Gamma(\nu)\xi^{\nu-\gamma-1}\int_{-\infty}^{\infty}\frac{(-\xi)^{-w}}{\Gamma(\nu-\gamma-w)}\times$$

$${}^{\alpha}C_{\gamma+w}\,{}_1F_1(\nu;\nu-\gamma-w;a\xi)D_x^{\alpha-\gamma-w}\delta(x-\xi)dw.$$

Example (4). If $f(x)=\left(1-x^2\right)^\nu$

$$\rho(x)=1-x,$$

then following Osler (1970, p. 668)

$$D_{1-x}^{\gamma+r+\nu}\left(1-x^2\right)^\nu\Big|_{x=\xi}=2^\nu\Gamma(\nu+1)\left(1-\xi^2\right)^{-\frac{1}{2}(\gamma+r)}P_\nu^{\gamma+r}(\xi),$$

Thus (2.5) becomes

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\left(1-x^2\right)^{\nu} D_{1-x}^{\alpha}\delta(x-\xi)=\sum_{r=-\infty}^{\infty} 2^{\nu}(-1)^{2\alpha-\gamma-r}\,{}^{\alpha}C_{\gamma+r}\,\Gamma(\nu+1)\times$$

$$\left(1-\xi^2\right)^{-\frac{1}{2}(\gamma+r)} P_{\nu}^{\gamma+r}(\xi) D_{1-x}^{\alpha-\gamma-r}\delta(x-\xi)$$

and (2.7) becomes

$$\left(1-x^2\right)^{\nu} D_{1-x}^{\alpha}\delta(x-\xi)=(-1)^{2\alpha-\gamma}\,2^{\nu}\Gamma(\nu+1)\int_{-\infty}^{\infty}(-1)^{-w}\,{}^{\alpha}C_{\gamma+w}\times$$

$$\left(1-\xi^2\right)^{-\frac{1}{2}(\nu+w)} P_{\nu}^{\gamma+w}(\xi) D_{1-x}^{\alpha-\gamma-w}\delta(x-\xi)\,dw$$

for all complex α and γ, except for negative integral α.

Acknowledgement. The author wishes to express his sincere thanks to Professor R.P. Agarwal, Department of Mathematics, Lucknow University for his valuable suggestions. The author is also grateful to the referee for his valuable remarks.

References

Ahuja, G. (1992) : On the product of an arbitrary function and the fractional derivative of Dirac-delta distribution. *Jour. Indian Math. Soc.* **58**, 231.

Erde' lyi, **A. et al.** (1955) : "*Tables of Integral transforms II*" McGraw Hill. New York.

Gelfand, I.M. and Shilov, G.E. (1964) : "*Generalized Functions. I*" Academic Press, New York,

Grunwald, A.K. (1867) : Ueber begrenzte Derivationen und deren Anwedung. *Z. Math. Phys.* **12** *441.*

Kanwal, R.P. (1983) : "*Generalized Functions : Theory and Technique*" Academic Press, New York.

Kober, H. (1940) : On fractional integrals and derivatives. *Quart. J. Math. Oxford.* **11**, 193.

Letnivov, A.V. (1868) : Theory of differentiation of fractional order. *Math. Sb.* **3**, 1.

Osler, T.J. (1970) : Leibnitz rule for fractional derivatives, generalized and an application to infinite series. *SIAM J. Appl. Math.* **18**, 658.

Oster, T. J. (1972) : The integral analogue of Leibnitz rule. *Math. Comp.* **26**. 903.

DEPARTMENT OF POST-GRADUATE
STUDIES IN MATHEMATICS
FEROZE GANDHI COLLEGE,
RAE BARELI

MAILING ADDRESS :—
127/1, CHANDER NAGAR
ALAMBAGH, LUCKNOW-226005

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc. **86**, 469-476 (1994)

ON GOOD EXTENSIONS OF SOME TOPOLOGICAL COVERING PROPERTIES

A. S. MASHHOUR[1], F. M. ZEYADA[2] AND A. H. ISMAIL[3]

(*Received 30 July 1993*)

Abstract : Following Lowen's approach to defining a good fuzzy compactness (Lowen 1978) and because of that the Chang-based dilutions are not good, in 1992, A. Bülbül and M.W. Warner introduced good extensions of almost compactness, near compactness and S-closedness. In 1986, A.S. Mashhour et al introduced the concepts of α-almost compactness, α-nearly compactness and αS-closedness and in 1989, K. M. Abd El-Hakeim introduced the concepts of α-semicompactness, α-pecompactness* and $\alpha\beta$-compactness. In this paper we aim to prove that the above concepts are good extensions. Moreover, some result have been obtained about these 'dilutions' of fuzzy compactness reflecting to some extent parallel properties in classical general topology.

1. Introduction : In Section 2, we give some necessary preliminaries. In Section 3, we prove that the concepts of α-almost compactness, α-nearly compactness and αS-cloedness introduced by A.S. Mashhour (1986); and also new six concepts, namely strong fuzzy almost compactness; weak fuzzy almost compactness; strong fuzzy nearly compactness; weak fuzzy nearly compactness; strong fuzzy S-closedness and weak fuzzy S-closedness, all are good extensions. The preserving of these concepts under some types of fuzzy continuous functions is studied. In Section 4, the concepts of α-semicompactness, α-precompactness, $\alpha\beta$-compactness introdced by K. M. Abd El-Hakeim (1989) are good extensions and additional good extensions, namely strong fuzzy semicompactness; strong fuzzy precompactness; strong fuzzy β-compactness; fuzzy semicompactness; fuzzy precompactness; fuzzy β-compactness; weak fuzzy β-compactness, are introduced. We preserve these concepts under some types of fuzzy continuous functions.

2. Preliminaries : Throughout this paper (X, δ) and (Y, σ) mean quasi fuzzy topological spaces (abbreviated as qfts's). The definitions of fuzzy sets, qfts's and other concepts can be found in (BülBül and Warner 1992, Chang 1968, Concilio and Gerla 1984, Pu and Liu 1980, Warren 1978, Weiss 1975 and Zadeh 1965). The definition of topologically generated fuzzy topological space in (Lowen 1976). The definition of the regularity of a qfts (X, δ) in (Concilio and Gerla 1984). The definition of α-almost compactness (resp α-nearly compactness, αS-closed) in (Mashhour et al 1986). The definition of Lo-almost compactness (resp Lo-nearly compactness, Lo-S-closed) in (BülBül and Warner 1992). The definition of almost fuzzy continuity (resp. fuzzy δ-continuity) in (Azad 1981) (resp. Ganguly and Saha 1988). The definition of fuzzy M-precontinuity (resp. fuzzy irresolute, fuzzy $M\beta$-continuity) in (Mashhour et al 1984) in (resp. (Mukherjee and Sinha 1989), (Abd El-Hakeim 1989)).

If $A \subset X$, then 1_A is a fuzzy set in X defined by :

---

* For convenience strong compactness [1,17] is renamed by precompactness.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

The fuzzy point $\alpha 1_a$ is just the fuzzy set $\alpha 1_{\{a\}}$, where $\alpha \in]0,1]$. The α (resp. α^*)-support for each member $A \in I^X$ is defined and denoted by :

$\alpha(A) = \{x \in X : A(x) > \alpha\}$, $\alpha \in I_1 = [0,1[$ (resp. $\alpha^*(A) = \{x \in X : A(x) \geq \alpha\}$, $\alpha \in I_o =]0,1]$) (Weiss 1975). If $G \in I^X$ then G is called semiopen (Abd EL-Monsef et al 1992) (resp. preopen (Fath-Alla 1984), β-open (Fath-Alla 1984), iff $G \subset G^{o-}$ (resp. $G \subset G^{-o}, G \subset G^{-o-}$). The concept of quasi coincident between two fuzzy sets A and B in X is defined and denoted by : AqB iff there exists a point $x \in X$ such that $A(x) + B(x) > 1$ (Pu and Liu 1980).

(X, δ) is said to be α-semicompact (Abd-El-Hakeim 1989) (resp. α-precompact (Abd El-Hkeim 1989), $\alpha\beta$-compct (Abd EL-Hakim 1989) iff each semiopen (resp. preopen, β-open) α-shading U of X has a finite α-subshading of X.

Note : Throughout this paper, we rename the Lo-almost compactness (resp. the Lo-nearly compactness, the Lo-S-closedness) of a qfts (X, δ) introduced in (BülBül and Warner 1992) by the fuzzy almost compactness (resp. the fuzzy nearly compactness, the fuzzy s-closedness).

3. Good extensions of almost compactness, nearly compactness and S-closedness.

DEFINITION : 3.1. *Let (X, δ) be a qfts.*

(1) *(X, δ) is said to be strong fuzzy almost compact (resp. strong fuzzy nearly compact, strong fuzzy S-closed) iff it is α-almost compact (resp. α-nearly compact, αS-closed) for each $\alpha \in I_1$.*

(2) *(X, δ) is said to be weak fuzzy almost compact (resp. weak fuzzy nearly compact, weak fuzzy S-closed) iff for every collection $\{G_j : j \in J\}$ of open (resp. open, semiopen) fuzzy sets in X such that $\bigvee_{j \in J} G_j = 1_X$ and for every $\varepsilon > 0$, there exists a finite subset F of J such that*

$$\bigvee_{j \in F} \overline{G}_j \geq (1-\varepsilon)1_X \left(\text{resp. } \bigvee_{j \in F} \overline{G}_j^o \geq (1-\varepsilon)1_X, \bigvee_{j \in F} \overline{G}_j \geq (1-\varepsilon)1_X \right).$$

THEOREM 3.1. *The implications contained in the following diagrams are valid for general qfts's :*

a – almost compactness ⇐ strong fuzzy almost compactness
⇓
weak fuzzy almost compactness ⇐ fuzzy almost compactnes

diagram (1)

a - nearly compactness ⇐ strong fuzzy nearly compactness
⇓
weak fuzzy nearly compactness ⇐ fuzzy nearly compactness

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

diagram (2)

$$aS\text{-}closed \Leftarrow strong\ fuzzy\ S\text{-}closed$$
$$\Downarrow$$
$$weakfuzzy\ S\text{-}closed \Leftarrow fuzzy\ S\text{-}closed$$

diagrm (3).

LEMMA 3.1. *Let (X, τ) be an ordinary topological space and $(X, W(\tau)$ the topogically generated fts of (X, τ). Let $\alpha \in I_1$, $\varepsilon > 0$ such that $0 < \alpha + \varepsilon < 1$ and $G \in I^X$. Then for each $x \in X$*

(1) $x \in \tau - cl\ (\alpha + \in (G))$ *implies* $W(\tau) - cl\ (G(x)) > \alpha$.

(2) $x \in \tau\text{-}int\ (\alpha^*\ (G))$ *implies* $W(\tau) - int\ (G)\ (x) \geq \alpha$.

(3) $x \in \tau\text{-}int\ (\tau - cl\ (\alpha + \varepsilon\ (G)))$ *implies* $W(\tau) - int\ (W(\tau) - cl(G)\ (x) > \alpha$.

Proof. (1) If $W(\tau) - cl(G)\ (x) \leq \alpha$, then $W(\tau) - cl(G)\ (x) < \alpha + \varepsilon$. Then there exists an $W(\tau)$ – closed fuzzy set H such that $H(x) < \alpha + \varepsilon$ and $H \geq G$. So $H'(x) > (a+\varepsilon)'$ and $H \geq G$. Then $x \notin \left((\alpha+\varepsilon)'(H')\right)'$ and $H' \leq G'$. Since $\left((\alpha+\varepsilon)'(H')\right)'$ is a τ-closed set containing $\alpha + \varepsilon\ (G)$ [Indeed, $y \in \alpha + \varepsilon(G) \Rightarrow G(y) > \alpha + \varepsilon \Rightarrow H'(y) \leq G'(y) < (\alpha+\varepsilon)' \Rightarrow y \notin (\alpha+\varepsilon)'(H') \Rightarrow y \notin \left((\alpha+\varepsilon)'(H')\right)'$.], then $x \notin \tau - cl\ (\alpha + \varepsilon(G))$.

(2) If $x \in \tau - \text{int}\ (\alpha^*\ (G))$, then there exists a τ-open set U such that $x \in U$ and $U \subset \alpha^*(G)$. So there exists an $W(\tau)$-open fuzzy set $\alpha 1_X \wedge 1_U$ such that $\alpha 1_x \in \alpha 1_X \wedge 1_U$ and $\alpha 1_X \wedge 1_U \leq G$. Thus $\alpha 1_x \in W(\tau) - \text{int}(G)$. Hence $W(\tau) - \text{int}(G)(x) \geq \alpha$.

(3) By (1) one can deduce that if $x \in \tau - cl\ (\alpha + \varepsilon(G))$, then $W(\tau) - cl(G)\ (x) \geq \alpha + \varepsilon_0$ for some $\varepsilon_0 > 0$, i.e. $\tau - cl(\alpha + \varepsilon(G)) \subset (\alpha + \varepsilon_o)^* (W(\tau) - cl(G))$. So $x \varepsilon \tau - int\ (\tau - cl\ (\alpha + \varepsilon(G))$ implies $x \in \tau - int\ ((\alpha + \varepsilon_0)^*\ (W(\tau) - cl(G)))$. Hence by (2) $W(\tau) - int\ (W(\tau) - cl(G))\ (x) \geq \alpha + \varepsilon_0 > \alpha$.

LEMMA 3.2. (Corollary 3.2.) (BülBül and Warner 1992). *Denoting by 1_A the characteristic function of a general subset A of X, we have $1_{\tau-cl(A)} = W(\tau) - cl(1_A)$ and $1_{\tau\text{-int}(A)} = W(\tau) - \text{int}(1_A)$.*

LEMMA 3.3. (1) *If G is $W(\tau)$ – semiopen (resp. $W(\tau)$ – preopen), $W(\tau) - \beta$ – open), then $\alpha(G)$ is τ – semiopen (resp. τ-preopen, $\tau - \beta$–open).*

(2) *If A is τ–semiopen (resp. τ-preopen, τ–β–open) then 1_A is $W(\tau)$–semiopen (resp. $W(\tau)$–preopen, $W(\tau)$–β–open).*

Proof. The proof of the statements involved the semiopening is obtained in Lemma 3.3 (BülBül and Warner 1992) and throughout the proof of Theorem 4.4 (BülBül and Warner 1992) The other statements can be proved directly from

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Lemma 3.1. and Lemma 3.2, for example if G is an $W(\tau)$–β–open fuzzy set, then $\alpha(G) \subseteq \alpha\Big(W(\tau) - cl\Big(W(\tau) - \text{int}\big(W(\tau) - cl(G)\big)\Big)\Big) \subseteq \tau - cl\Big(\alpha\Big(W(\tau) - \text{int}\big(W(\tau) - cl(G)\big)\Big)\Big) \subseteq \tau - cl\Big(\tau - \text{int}\Big(\tau - cl\big(\alpha(G)\big)\Big)\Big)$ and hence $\alpha(G)$ is τ–β–open. Also if A is a τ–preopen, then $1_A \leq 1_{\tau-cl\left(\tau-\text{int}(A)\right)}$ $\leq W(\tau) - cl\big(W(\tau) - \text{int}(1_A)\big)$ and hence 1_A is $W(\tau)$–preopen.

THEOREM 3.2. *The concepts of strong fuzzy almost compactness, weak fuzzy almost compactness and α-almost compactness are good extensions of the topological almost compactness property.*

Proof. In regard to the digram (1) Theorem 3.1 it suffices to prove the following statements :

(i) (X, τ) is almost compact $\Rightarrow$ $(X, W(\tau))$ is strong fuzzy almost compact ;

(ii) $(X, W(\tau))$ is weak fuzzy almost compact $\Rightarrow$ (X, τ) is almost compact; and

(iii) $(X, W(\tau))$ is α–almost compact $\Rightarrow$ (X, τ) is almost compact, where (X, τ) is an ordinary topological space.

For providing (i), let $U \subset W(\tau)$ be an $W(\tau)$–open α–shading of X and $\alpha \in I_1$. Then for every $x \in X$, there exists $G_x \in U$ such that $G_x(x) > \alpha$. So there exists $\varepsilon_x \in I_0$ such $G_x(x) > \alpha + \varepsilon_x > \alpha$. Thus the collection $\{\alpha + \varepsilon_x(G_x) : x \in X\}$ is a τ-open cover of X. Since X is almost compact, then there exists a finite subfamily A of X such that the collection $\{\tau - cl(\alpha + \varepsilon_x(G_x)) : x \in A\}$ is a τ–cover of X. Then from Lemma 3.1 (1) the collection $\{W(\tau)$-$cl(G_x) : x \in A\}$ is an α-shading of X. Hence $(X, W(\tau))$ is strong fuzzy almost compact.

For providing (ii), let $V \in \tau$ be a τ-open cover of X. Then $\underset{A \in V}{\vee} 1_A = 1_X$, where 1_A is the characteristic function of A. Choose $\varepsilon \in]0, 1[$, then it follows from the weak fuzzy almost compactness of $(X, W(\tau))$ that there exists a finite subfamily $V_o \subset V$ such that $\underset{A \in V_o}{\vee} W(\tau) - cl(1_A) \geq (1-\varepsilon)1_X$. Since $1_{\tau-cl(A)} \geq W(\tau) - cl(1_A)$ for every $A \in V$ and $(1-\varepsilon)1_X(x) > 0$, then $\underset{A \in V_o}{\vee} 1_{\tau-cl(A)}(x) > 0$. So for every $x \in X$, there exists $A \in V_0$ such that $x \in \tau$–$cl(A)$. Thus the collection $\{\tau - cl(A) : A \in V_o\}$ is a cover of X. Hence (X, τ) is almost compact.

For proving (iii) , let $V \subset \tau$ be a τ–open cover of X. Then $\{1_A : A \in V\}$ is a $W(\tau)$–open α-shading of X. Since $(X, W(\tau))$ is α–almost compact, then there exists a finite subfamily V_o of V such that $\underset{A \in V_o}{\vee} W(\tau) - cl(1_A) > \alpha 1_X$. Thus the collection $\{\tau$–$cl(A) : A \in V_0\}$ is cover of X. Hence (X, τ) is almost compact.

THEOREM 3.3. *The concepts of strong fuzzy nearly compactness, weak fuzzy nearly conpactness and α–nearly compactness are good extension of the topological nearly compactness property.*

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. From Theorem 3.1 (diagram (2)), it is enough to prove the following statements :

(i) (X, τ) is nearly compact $\Rightarrow$ $(X, W(\tau))$ is strong fuzzy nearly compact; (ii) $(X, W(\tau))$ is weak fuzzy nearly compact $\Rightarrow$ (X, τ) is nearly compact; and (iii) $(X, W(\tau))$ is α–nearly compact $\Rightarrow (X, \tau)$ is nearly compact, where (X, τ) is an ordinary topological space.

For proving (i) proceeding again as in the corresponding statement (i) in the proof of Theorem 3.2, we obtain a finite subfamily A of X such that the collection $\left\{\tau\text{-}int\left(\tau\text{-}cl\left(\alpha+\varepsilon_x(G_x)\right)\right) : x \in A\right\}$ covers X. So by Lemma 3.1 (3), one can deduce that the collection $\left\{W(\tau)-\text{int}\left(W(\tau)-cl(G_x)\right) : x \in A\right\}$ is an α-shading of X.

The proof of the statements (ii) and (iii) follows closely the argument of the corresponding statements (ii) and (iii) in the proof of Theorem 3.2, since from Lemma 3.2 one can have

$$W(\tau)-\text{int}\left(W(\tau)-cl(1_A)\right) = 1_{\tau-\text{int}(\tau-cl(A))} \text{ for any general subset } A \text{ of } X.$$

THEOREM 3.4. *The concepts of strong fuzzy S-closedness, weak fuzzy S-closedness and αS-closedness are good extension of the topological S-closedness property.*

Proof. From Theorem 3.1 (diagram (3)), it suffices to prove the following statements :

(i) (X, τ) is S-closed $\Rightarrow (X, W(\tau))$ is strong fuzzy S-closed;

(ii) $(X, W(\tau))$ is weak fuzzy S-closed $\Rightarrow (X, \tau)$ is S-closed; and

(iii) $(X, W(\tau))$ is αS-closed$\Rightarrow (X, \tau)$ is S-closed, where (X, τ) is an ordinary topological space.

The proof of these statements (i) – (iii) is exactly similar to that of the corresponding statements (i) – (iii) in Theorem 3.2, since from Lemma 3.3 we have : (1) for each $W(\tau)$ – semiopen fuzzy set G, $\alpha(G)$ is τ-semiopen and (2) for each τ-semiopen set A, 1_A is $W(\tau)$ – semiopen.

THEOREM 3.5. *Let (X, δ) be a qfts.*

(1) *If X is regular and strong (resp. weak) fuzzy almost compact, then X is strong (resp. weak) fuzzy compact.*

(2) *If X is FED and strong (resp. weak) fuzzy almost compact, then X is strong (resp. weak) fuzzy S-closed.*

THEOREM 3.6. *If (X, δ) is a regular FED-qfts, then the following are equivalent :*

(1) *X is strong (resp. weak) fuzzy compact.*

(2) *X is strong (resp. weak) fuzzy nearly compact.*

(3) *X is strong (resp. weak) fuzzy almost compact.*

(4) *X is strong (resp. weak) fuzzy S-closed.*

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEFINITION 3.2. *A function f from a qfts (X, δ) into a qfts (Y, σ) is said to be fuzzy θ-irresolute iff for each fuzzy point* $\alpha 1_x$ *in X and for any semiopen fuzzy set U in Y with* $Uqf(\alpha 1_x)$, *there exists a semiopen fuzzy set G in X with* $Gq\alpha 1_x$ *such that* $f(\overline{G}) \leq \overline{U}$.

THEOREM 3.7. *Let f be a subjective function from a qfts (X, δ) into a qfts (Y, σ). Then*

(1) *if f is almost fuzzy continuous and (X, δ) is strong fuzzy (resp. fuzzy, weak fuzzy) almost compact, then so is (Y, σ).*

(2) *if f is fuzzy δ-continous and (X, δ) is strong fuzzy (resp. fuzzy, weak) nearly compact, then so is (Y, σ).*

(3) *if f is fuzzy θ-irresolute and (X, δ) is strong fuzzy (resp. fuzzy, weak fuzzy) S-closed, then so is (Y, σ).*

4. Goods extensions of Semicompactness, Precompactness and β-Compactness.

First one can deduce the following theorem by making use of Lemma 3.3.

THEOREM 4.1. *The concept of α-semicompactness (Abd EL-Hakeim 1989) (resp. α-precompactness) (Abd EL-Hakeim 1989), αβ-compactness (Abd EL-Hakeim 1989) is good extension of the topological semicompactness (Dorsett 1981) (resp. precompactness) (Mashhour, Abd El-Monsef, Hasanein and Noiri 1984), β-compactness (Abd-EL-Monsef and Kozae 1985) property.*

DEFINITION 4.1. *Let (X, δ) be a qfts.*

(1) *(X, δ) is said to be strong fuzzy semicompact (resp. strong fuzzy precompact, strong fuzzy β-compact) iff it is α-semicompact (resp. α-precompact, αβ-compact) for each* $\alpha \in I_f$.

(2) *(X, δ) is said to be fuzzy semicompact (resp. fuzzy precompact, fuzzy β-compact) iff for each family β of semiopen (resp. preopen, β-open) fuzzy sets in X and for each* $\alpha \in I$ *such that* $\sup_{\mu \in \beta} \mu \geq \alpha 1_X$ *and for each* $\varepsilon \in]0, \alpha]$, *there exists* $\beta_o \in 2^{(\beta)}$ *such that* $\sup_{\mu \in \beta_o} \mu \geq (\alpha - \varepsilon) 1_X$.

(3) *(X, δ) is said to be weak fuzzy semicompact (resp. weak fuzzy precompact, weak fuzzy β-compact) if for each family β of semiopen (resp. preopen, β-open) fuzzy sets in X such that* $\sup_{\mu \in \beta} \mu = 1_X$ *and for each* $\varepsilon > 0$, there exists $\beta_0 \in 2^{(\beta)}$ such that $\sup_{\mu \in \beta_0} \mu \geq (1 - \varepsilon) 1_X$.

Also Lemma 3.3 is useful for proving the following three theorems.

THEOREM 4.2. *The concepts of strong fuzzy semicompactness, weak fuzzy semicompactness and fuzzy semicompactness are good extensions of the topological semicompactness property.*

THEOREM 4.3. *The concepts of strong fuzzy precompactness, weak fuzzy precompactness and fuzzy precompactness are good extensions of the topological precompactness property.*

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

THEOREM 4.4. *The concepts of strong fuzzy β-compactness, weak fuzzy β-compactness and fuzzy β-compactness are good extensions of the topological β-compactness property.*

The preserving of the concepts in Definition 4.1 under some types of fuzzy continuous functions is given by the following theorem :

THEOREM 4.5. *Let f be a subjective function from a qfts (X, δ) into a qfts (Y, σ). Then*

(1) *if f fuzzy M-precontinuous and (X, δ) is strong fuzzy (resp. fuzzy, weak fuzzy) precompact, then so is (Y, σ).*

(2) *if f is fuzzy irresolute and (X, δ) is strong fuzzy (resp. fuzzy, weak fuzzy) semicompact, then so is (Y, σ).*

(3) *if f is fuzzy Mβ-continuous and (X, δ) is strong fuzzy (resp. fuzzy, weak fuzzy) β-compact, then so is (Y, σ).*

References

Abd El-Hakeim, K.M., (1989) : On fuzzy topological spaces, *Ph. D. Thesis*, Assiut Univ.

Abd El-Monsef, M. E. and Kozae, A. M. (1985) : Some generalized forms of compactness and closedness, *Delta J. Sci.* **(9)** (2), 254.

Abd El-Monsef, M. E. Zeyada, F. M., El-Deeb, S. N. and Hanafy, I. M. (1992) : Good extensions of paracompactness, *Math. Japonica* **37**, 1,000.

Azad, K. K. (1981) : On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity, *J. Math. Anal. Appl.* **82**, 14.

Bülbül, A. and Warner, M. W. (1992) : Some good dilutions of fuzzy compactness. *Fuzzy Sets and Systems* **51**, 111.

Chang, C.L. (1968) : Fuzzy topological spaces, *J. Math. Anal. Appl.* **24**, 182.

Di Concilio, A. and Gerla, G. (1984) : Almost compactness in fuzzy topological spaces, *Fuzzy Sets and Systems* **13**, 187.

Dorsett, Ch. (1981) : Semiconvergence and semicompactness, *Indian J.M.M.* **XIX**, 1.

Gantner; T. E., Steinlage R. C. and Warren, R. W. (1978) : Compactness in fuzzy topological spaces, *J. Math. Anal. Appl.* 62, 547.

Fath-Alla, M. A. (1984) : On fuzzy topological spaces, *Ph. D. Thesis*, Assiut Univ.

Ganguly S. and Saha, S. (1988) : A note on δ-connected sets in fuzzy set theory, *Simon Stevin* **62**, 127.

Lowen, R. (1976) : Fuzzy topological spaces and fuzzy compactness, *J. Math. Anal. Appl.* **56**, 621.

Lowen, R. (1978) : A comparison of different compactness notions in fuzzy topological spaces, *J. Math. Anal. Appl.* **64**, 446.

Mashhour, A. S., Abd El-Monsef M. E. Hasanein, I. A. and Noiri, T. (1984) : Strongly compact spaces, *Delta J. Sci.* **8(1)**, 30.

Mashhour A. S., Ghanim M. H. and Fath-Alla, M. A. (1986) : α-separation axioms and α-compactness in fuzzy topological spaces, *Rocky Mount., J. Math.* **16(3)**, 591.

Mukherjee, M. N. and Sinha, S. P. (1989) : Irresolute and almost open functions between fuzzy topological spaces, *Fuzzy Sets and Systems* **29**, 381.

Nanda, S. (1991) : Strongly compact fuzzy topological spaces, *Fuzzy sets and Systems* **42**, 259.

Pu, P. M. and Liu, Y. M. (1980) : Fuzzy topology : I : Neighbourhood structure of a fuzzy point Moore-Smith convergence, *J. Math. Anal.* **76**, 571.

Warren, R. H. (1978) : Neighborhoods, bases and continuity in fuzzy topological spaces, *Rocky Mount. J. Math.* **8**, 459.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Weiss, M. D. (1975) : Fixed points, separation and induced topologies for fuzzy sets, *J. Math. Anal. Appl* **50**, 142.

Zadeh, L. A. (1965) : Fuzzy sets, *Information and Control*, **8**, 338.

[1]DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE,
ASSIUT UNIVERSITY,
ASSWAN, EGYPT

[2]DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE,
SUEZ CANAL UNIVERSITY,
ISMAILIA, EGYPT

[3] DEPARTMENT OF
MATHEMATICS,
FACULTY OF SCIENCES,
ASSIUT UNIVERSITY,
SOHAG, EGYPT

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

THE COUNCIL OF THE CALCUTTA MATHEMATICAL SOCIETY 1994—1996

President

M. Dutta

Vice-Presidents

T. K. Mukherjee, P. P. Chatterjee, L. Debnath
S.P. Pal, K. R. Ramachandran.

Treasurer

U. Basu

Secretary

R.N. Sen

Editorial Secretary

C. G. Chakrabarti

Assistant Secretary

Sulekha Mukherjee

Other Members of the Council

B. C. Chakraborty, N. C. Ghosh, A. B. Roy, N. D. Sengupta,
M. R. Adhikari, P. K. Chaudhuri, D. K. Basu, M. Tarafdar,
P. K. Das, B. K. Sengupta, R. N. Chakrabarti, L. M. Saha.

All correspondence with the society, (Subscriptions to the Bulletin, admission fees and annual contributions of members etc.) are to be made with :

The Secretary

Calcutta Mathematical Society

"Asutosh Bhavan', AE-374, Sector-I Calcutta 70 064 (INDIA)

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CALCUTTA MATHEMATICAL SOCIETY

AE-374 SECTOR I, SALT LAKE, CALCUTTA-700 064, INDIA

NOTES FOR AUTHORS

On General Requirements

1. The Bulletin of the Calcutta Mathematical Society publishes papers presented to the society containing original works in different branches of mathematics and allied sciences.

2. Authors should send to the *Editorial Secretary* of the Society two copies of their communications (one original copy) typed neatly on white paper, on one side only, with double spacing and with a margin of 4 c.m. together with all ancillary materials, while a further copy should be retained by them.

3. Authors will receive ordinarily one proof corrections. Corrections to the proofs should be restricted to printer's errors only and made according to standard practice, Cost of any other corrections will be charged to the authors.

4. Authors are required to inform the Editorial Secretary of any change of address which occurs whilst their papers are in the process of publication.

5. The printing cost of the paper at the rate of Rs. 60/- or $ 20 per page should be paid by the Authors or their Institutions.

On Script Requirement

1. The first page should bear in order—

The title, brief but informative; the name(s) of the authors(s); the address to which the proofs should be sent; a running title of not more than 55 letters or signs; a brief abstract followed by (i) Major field(s) (ii) AMS Sunject classification and the introduction.

2. Equations and synbols may be handwritten or typewritten; clearity for the printer is essential. Unusual symbols should be identified by name in the margin (in pencil) the first time they are used. If handwritten, care should be taken to distinguish between capital and lower case letters *e.g. c, k, o, p, s, w*, phi, psi; between alpha (α) and a; kappa (κ) and k; mu (μ) and u; nu (ν) and v; eta (η) and n; omega (ω) and w. When used as order symbol o and 0 should be shown as $\underline{o}$ and $\underline{\underline{O}}$.

Fractional exponents are to be used instead of root sign, n! for $\lfloor n$. exp ( ) notation when the exponent if complicated. Repetition of comlicated expression is to be avoided by representing it with a symbol.

3. Reference to the literature in the text should be by author's surname and year in parentheses. References cited in the text should be listed at the end of the paper in the alphabetical order of the surname of the author in the following way (a recent number of Bulletin may be a proper guide) :

Chandrasekhar, S. (1961) Hydrodynamic and Hydromagnetic Stability, Oxford University Press, London,p. 10.

Wilson, B.M. (1992) Proc. Lond. Math. Soc. (2), **21**, 235.

4. Tables should be typed on pages separate from the text, numbered sequentially and planned to fit the page width of the journal (13 c.m.). Table numbers should be typed separately in sequence on the manuscript paper.

5. Line drawing must be clearly and sharply drawn (duly traced by a draughtsman) with black India ink on white tracing paper. Figures should be drawn about 4 times the size which would appear in print. Standard size after reduction is normaly 7 c.m. Each figure must have a number. Figure numbers should be typed in sequences separately on manuscript paper. The back of each figure should bear the name(s) of author(s) and the figure number.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc., *86*, 477–480 (1994)

A CLASS OF BILATERAL GENERATING FUNCTIONS OF MODIFIED BESSEL POLYNOMIALS

M. C. MUKHERJEE AND A. K. CHONGDAR

(*Received 5 September 1988*)

Abstract. In this note a general theorem in connection with the derivation of bilateral generating relation of modified Bessel polynomials $\left\{Y_n^{(\alpha+n)}(x)\right\}$ has been derived from the Lie group view point.

1. Introduction. Recently modified Bessel polynomials $\left\{Y_n^{(\alpha+n)}(x)\right\}$, where $Y_n^{(\alpha)}(x)$ is the Bessel polynomials given by Krall and Frink (1949), has been extensively studied by Chongdar (1984, 1987), Guhathakurta and Chongdar (1984) by using Weisner's group theoretic method (Weisner 1955) in connection with the derivation of generating functions.

Very recently (Mukherjee and Chongdar 1987), the present authors, have studied the modified Bessel polynomials defined by

$$Y_n^{(\alpha+n)}(x) = {}_2F_0\left(-n, 2n+\alpha-1; -; \frac{X}{\beta}\right) \tag{1.1}$$

by Lie-algebraic method introduced by Weisner and derived a large number of new generating functions.

The aim at presenting this note is to establish the following theorem on the derivation of class of bilateral generating functions of $Y_n^{(\alpha+n)}(x)$ from the Lie group view point.

THEOREM. *If there exists a generating relation of the form*

$$G(x,w) = \sum_{n=0}^{\infty} a_n Y_n^{(\alpha+n)}(x) w^n \tag{1.2}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

then

$$(1-wx)^{1-\alpha}\exp(w\beta)G\left(\frac{x}{1-wx},\frac{wy}{(1-wx)^2}\right)=\sum_{m=0}^{\infty}w^m g_m(y)Y_m^{(\alpha+2n-m)}(x)$$

where

$$g_m(y)=\sum_{n=0}^{m}a_n\frac{\beta^{m-n}}{(m-n)!}y^n.$$

The importance of this theorem lies in the fact that whenever one knows a generating relation of the form (1.2), the corresponding bilateral generating relation can at once be written down from the relation (1.3). Thus, we see that one can get a large number of generating relations from (1.3) by attributing different suitable values to a_n in (1.2).

2. Proof of the Theorem. For the modified Bessel polynomial $Y_n^{(\alpha+n)}(x)$ let us consider the partial differential operator $\mathbb{R}$ (Mukherjee and Chongdar 1987) as follows:

$$\mathbb{R}=x^2yz^{-2}\frac{\partial}{\partial x}+2xy^2z^{-2}\frac{\partial}{\partial y}+xyz^{-1}\frac{\partial}{\partial z}+(\beta-x)yz^{-2} \tag{2.1}$$

such that

$$\mathbb{R}\left(Y_n^{(\alpha+n)}(x)y^nz^\alpha\right)=\beta\,Y_{n+1}^{(\alpha+n-1)}(x)y^{n+1}z^{\alpha-2}.$$

The extended form of the group generated by $\mathbb{R}$ is

$$e^{w\,\mathbb{R}}f(x,y,z)=\left(1-xyz/z^2\right)\exp\left(w\beta y/z^2\right)\times$$
$$f\left(\frac{x}{(1-wxy/z^2)},\frac{y}{(1-wxy/z^2)^2},\frac{z}{(1-wxy/z^2)}\right)$$

Let us assume the generating relation

$$G(x,w)=\sum_{n=0}^{\infty}a_nY_n^{(\alpha+n)}(x)w^n. \tag{2.4}$$

Replacing w by wy and then multiplying both sides by z^α we get

$$z^\alpha G(x,wy)=\sum_{n=0}^{\infty}a_nw^n\left(Y_n^{(\alpha+n)}(x)y^nz^\alpha\right).$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Operating both sides of (2.5) by exp $(w\,\mathrm{IR})$ we get

$$\exp(w\,\mathrm{IR})z^{\alpha}G(x,wy) = \exp(w\,\mathrm{IR})\sum_{n=0}^{\infty} a_n w^n \left(Y_n^{(\alpha+n)}(x)y^n z^{\alpha}\right). \tag{2.6}$$

Now, from the left member of (2.6) we get

$$\exp(w\,\mathrm{IR})z^{\alpha}G(x,wy) = (1-wxy/z^2)^{1-\alpha}\exp\left(w\beta y/z^2\right)\times$$
$$z^{\alpha}G\left(\frac{x}{(1-wxy/z^2)},\frac{wy}{(1-wxy/z^2)^2}\right). \tag{2.7}$$

Also from the right member of (2.6) we have

$$\exp(w\,\mathrm{IR})\sum_{n=0}^{\infty} a_n w^n Y_n^{(\alpha+n)}(x)y^n z^{\alpha}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n \frac{w^{n+m}}{m!}\beta^m Y_{n+m}^{(\alpha+n-m)}(x)y^{n+m}z^{\alpha-2m}$$
$$= z^{\alpha}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n \frac{(wy)^{m+n}}{m!}(\beta/z^2)^m Y_{n+m}^{(\alpha+n-m)}(x)$$
$$= z^{\alpha}\sum_{m=0}^{\infty}\sum_{n=0}^{m} a_n \frac{(w\beta y/z^2)^m}{(m-n)!}(\beta/z^2)^{-n} Y_m^{(\alpha+2n-m)}(x). \tag{2.8}$$

Equating (2.7) and (2.8) and then putting $y/z^2 = 1$, we get

$$(1-wx)^{1-\alpha}\exp(w\beta)G\left(\frac{x}{1-wx},\frac{wy}{(1-wx)^2}\right) = \sum_{m=0}^{\infty} w^m g_m(y) Y_m^{(\alpha+2n-m)}(x),$$

where

$$g_m(y) = \sum_{n=0}^{m} a_n \frac{\beta^{m-n}}{(m-n)!}y^n.$$

This completes the proof of the theorem.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

References

Chongdar, A. K. (1984) : Some generating functions of modified Bessel polynomials from the view point of Lie group, *Internat. Jour. Math. and Math. Sci., 7,* 823.

Mukherjee, M. C. and **Chongdar, A. K.** (1987) : Lie algebra and modified Bessel polynomials, *Bull. Inst. Math. Acad. Sinica, 15(4),* 391.

Guhathakurta, B. K. and **Chongdar, A. K.** (1982) : Some generating functions of modified Bessel polynomials by Lie algebraic method, *Bull. Inst. Math. Acad. Sinica, 14(2),* 215.

Krall, H. L. and **Frink, O.** (1949) : A new class of orthogonal polynomials : the Bessel polynomials, *Trans. Amer. Math. Soc. 65,* 100.

Mukherjee, M. C. and **Chongdar, A. K.** (1987) : Derivation of some generating functions of modified Bessel polynomials by Lie algebraic method, *Bull. Cal. Math. Soc. 80,* 411.

Weisner, L. (1955) : Group-theoretic origin of certain generating functions, *Pacific Jour. Math., 5,* 1033.

NETAJINAGAR VIDYAMANDIR
CALCUTTA 700092
INDIA

AND

DEPARTMENT OF MATHEMATICS
BANGABASI EVENING COLLEGE
19 R K CHAKRABORTY SARANI
CALCUTTA 700009
INDIA

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc., 86, 481–490 (1994)

ON THE WELL-POSEDNESS OF A DISSIPATIVE RLW CLASS OF EQUATIONS

H. M. EL-SHERBINY AND M. A. ABDEL-RAZEK

(*Received 7 March 1991*)

Abstract. A class of nonlinear partial differential equations containing the RLW equation is presented. Its corresponding initial value problem is investigated. Existence and uniqueness theorems of the general solution are given via the use of the fixed point theorem and contraction mappings on Banach spaces.

1. Introduction. This paper is concerned with the investigation of the initial value problem corresponding to a new class of non-linear dispersive dissipative differential equations.

$$u_t + u_x + a_1 u^n u_x + a_3 u_{xx} + a_4 u_{xt} + a_7 u_{xxt} = 0, \quad -\infty < x < \infty, \quad t \geq 0, \tag{1.1}$$
$$u(x,0) = f(x),$$

where $a_i (i = 1, 3, 4, 7)$ and $n > 0$ are real and constants with $a_7 < 0$ and $u(x,t)$ is a scalar field defined for all $(x,t) \in \mathbb{R} \times [0,\infty)$. This class is related to the regularized long wave (RLW) equation (if $a_1 = n = -a_7 = 1$ and $a_3 = a_4 = 0$) (Benjamine, Bona and Mahony 1972).

$$u_t + u_x + uu_x - u_{xxt} = 0. \tag{1.2}$$

In (El-Sherbiny, 1972) it has been proved that if the algebraic equation

$$b_6 - b_7\lambda + b_8\lambda^2 - b_9\lambda^3 = 0, \tag{1.3}$$

has simple real root and one multiple root of order two, then the class of differential equations

$$u_t + u_x + b_1 u^n u_x + b_2 u^n u_t + b_3 u_{xx} + b_4 u_{xt} + b_5 u_{tt} + b_6 u_{xxx} + b_7 u_{xxt} + b_8 u_{xtt} + b_9 u_{ttt} = 0, \tag{1.4}$$

can be reduced, equivalently, to

$$u_t + u_x + a_1 u^n u_x + a_2 u^n u_t + a_3 u_{xx} + a_4 u_{ut} + a_5 u_{tt} + a_7 u_{xxt} = 0. \tag{1.5}$$

Thus, investigating the well-posedness of (1.5), satisfying the condition (1.3) is inherited to that for (1.5). Thus, the importance of investigating the existence and uniqueness of the solution of the initial value problem (1.1) is for two reasons, firstly in building a theory of well-posedness of the initial value problem corresponding to (1.4) and secondly in generalizing the study of the RLW equation which has been the subject of interest by a number of authors such as (El-Sherbiny 1987, El-Sherbiny, Hussein and Moustafa 1990).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

2. The Integral Form of the Solution. Rewriting the class (1.1) in the form

$$\left(\frac{\partial^2}{\partial x^2}+\frac{a_4}{a_7}\frac{\partial}{\partial x}+\frac{1}{a_7}\right)u_t=-\frac{1}{a_7}\frac{\partial}{\partial x}\left(u+\frac{a_1}{n+1}u^{n+1}+a_3u_x\right), \tag{2.1}$$

which can be regarded as ordinary differential equation in u_t. By using the Green's function of the operator $\left(\frac{\partial^2}{\partial x^2}+\frac{a_4}{a_7}\frac{\partial}{\partial x}+\frac{1}{a_7}\right)$, then (2.1) can be inverted to

$$u_t=-\frac{1}{a_7}\int_{-\infty}^{\infty}G(x,\eta)\left\{u(\eta,t)+\frac{a_1}{n+1}u^{n+1}(\eta,t)+a_3u_\eta(\eta,t)\right\}\delta n \tag{2.2}$$

where $G(x,\eta)$ is the Green's function of the operator $\left(\frac{\partial^2}{\partial x^2}+\frac{a_4}{a_7}\frac{\partial}{\partial x}+\frac{1}{a_7}\right)$ and defined by (see Appendix A)

$$G(x,\eta)=\begin{cases}\dfrac{-a_7}{\sqrt{a_4^2-4a_7}}e^{\beta(x-\eta)}, & x\le\eta\\[2ex] \dfrac{-a_7}{\sqrt{a_4^2-4a_7}}e^{\alpha(x-\eta)}, & x\ge\eta\end{cases} \tag{2.3}$$

with

$$\alpha=\frac{1}{2a_7}\left[-a_4+\sqrt{a_4^2-4a_7}\right]<0 \quad\text{and}\quad \beta=\frac{1}{2a_7}\left[-a_4-\sqrt{a_4^2-4a_7}\right]>0$$

for all $a_4\in(-\infty,\infty)-\{0\}$ and $a_7<0$.

Inserting (2.3) into (2.2) we have

$$\begin{aligned}u_t=&\frac{\alpha}{\sqrt{a_4^2-4a_7}}\int_{-\infty}^{x}e^{\alpha(x-\eta)}\left[(1+a_3\alpha)u\frac{a_1}{n+1}u^{n+1}\right]d\eta\\&+\frac{\beta}{\sqrt{a_4^2-4a_7}}\int_{x}^{\infty}e^{\beta(x-\eta)}\left[(1+a_3\beta)u+\frac{a_1}{n+1}u^{n+1}\right]d\eta+\frac{a_3}{a_7}u.\end{aligned} \tag{2.4}$$

Integrating (2.4) once with respect to t and using the initial condition $u(x,0)=f(x)$ we have

$$\begin{aligned}u(x,t)=f(x)&+\frac{\alpha}{\sqrt{a_4^2-4a_7}}\int_0^t\int_{-\infty}^{x}e^{\alpha(x-\eta)}\left[(1+a_3\alpha)u(\eta,\tau)+\frac{a_1}{n+1}u^{n+1}(\eta,\tau)\right]d\eta d\tau\\&+\frac{\beta}{\sqrt{a_4^2-4a_7}}\int_0^t\int_{x}^{\infty}e^{\beta(x-\eta)}\left[(1+a_3\beta)u(\eta,\tau)+\frac{a_1}{n+1}u^{n+1}(\eta,\tau)\right]d\eta d\tau\\&+\frac{a_3}{a_7}\int_0^t u(x,\tau)d\tau.\end{aligned} \tag{2.5}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Hence, the original initial value problem (1.1) is formally equivalent to an integral equation and we can now rewrite this in the form

$$u = Au = Bu + f(x) \tag{2.5b}$$

where,

$$\begin{aligned} Bu = {} & \frac{\alpha}{\sqrt{a_4^2 - 4a_7}} \int_0^t \int_{-\infty}^x e^{\alpha(x-\eta)} \left[(1 + a_3\alpha)u(\eta,\tau) + \frac{a_1}{n+1} u^{n+1}(\eta,\tau) \right] d\eta d\tau \\ & + \frac{\beta}{\sqrt{a_4^2 - 4a_7}} \int_0^t \int_x^{\infty} e^{\beta(x-\eta)} \left[(1 + a_3\beta)u(\eta,\tau) + \frac{a_1}{n+1} u^{n+1}(\eta,\tau) \right] d\eta d\tau \\ & + \frac{a_3}{a_7} \int_0^t u(x,\tau) d\tau. \end{aligned}$$

Having obtained the integral form (2.5) of the general solution of (1.1), we turn to study the well-posedness of the problem.

3. Existence. The proof of existence of a solution of (1.1) is reduced to that of the integral equation (2.5). For this purpose let $\ell_{t_\circ}$ be the space of all continuous and bounded functions defined on $\mathbb{R} \times [0, t_\circ]$ with norm defined by $\|.\| = \sup\limits_{\substack{x \in \mathbb{R} \\ 0 \le t \le t_\circ}} |.|$. It is clear that the space $\ell_{t_\circ}$ is a Banach space.

Consider now the integral equation (2.5), where the integral operator A acts on $\ell_{t_\circ}$, i.e., $A : \ell_{t_\circ} \to \ell_{t_\circ}$. Then for any $u_1, u_2 \in \ell_{t_\circ}$ and any $x \in \mathbb{R}, t \in [0, t_\circ]$;

$$\begin{aligned} |Au_1 - Au_2| &= |Bu_1 - Bu_2| \\ &\le \frac{|\alpha|}{\sqrt{a_4^2 - 4a_7}} \left| \int_0^t \int_{-\infty}^x e^{\alpha(x-\eta)} \left[(1 + a_3\alpha)(u_1 - u_2) + \frac{a_1}{n+1} (u_1^{n+1} - u_2^{n+1}) \right] d\eta d\tau \right| \\ &\quad + \frac{|\beta|}{\sqrt{a_4^2 - 4a_7}} \left| \int_0^t \int_x^{\infty} e^{\beta(x-\eta)} \left[(1 + a_3\beta)(u_1 - u_2) + \frac{a_1}{n+1} (u_1^{n+1} - u_2^{n+1}) \right] d\eta d\tau \right| \\ &\quad + \left| \frac{a_3}{a_7} \right| \left| \int_0^t (u_1 - u_2) d\tau \right| \\ &\le \frac{-\alpha}{\sqrt{a_4^2 - 4a_7}} \sup_{\substack{x \in \mathbb{R} \\ t \in [0, t_\circ]}} |u_1 - u_2| \left\{ |1 + a_3\alpha| + \frac{|a_1|}{n+1} \sup_{\substack{x \in \mathbb{R} \\ t \in [0, t_\circ]}} \left| \sum_{i=0}^n (u_1^{n-i} u_2^i) \right| \right\} \\ &\qquad \int_0^t \int_{\infty}^x e^{\alpha(x-\eta)} d\eta d\tau \\ &\quad + \frac{\beta}{\sqrt{a_4^2 - 4a_7}} \sup_{\substack{x \in \mathbb{R} \\ t \in [0, t_\circ]}} |u_1 - u_2| \left\{ |1 + a_3\beta| + \frac{|a_1|}{n+1} \sup_{\substack{x \in \mathbb{R} \\ t \in [0, t_\circ]}} \left| \sum_{i=0}^n (u_1^{n-i} u_2^i) \right| \right\} \\ &\qquad \int_0^t \int_x^{\infty} e^{\beta(x-\eta)} d\eta d\tau + \left| \frac{a_3}{a_7} \right| \sup_{\substack{x \in \mathbb{R} \\ t \in [0, t_\circ]}} |u_1 - u_2| \int_0^t d\tau \end{aligned}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$= \sup_{\substack{x\in \mathbb{R}\\ t\in[0,t_o]}} |u_1-u_2|\left\{\frac{1}{\sqrt{a_4^2-4a_7}}\left[|1+a_3\alpha|+|1+a_3\beta|+\frac{2|a_3|}{n+1}\right.\right.$$

$$\left.\left.\sup_{\substack{x\in \mathbb{R}\\ t\in[0,t_o]}}\left|\sum_{i=0}^{n}(u_1^{n-i}u_2^i)\right|\right]+\left|\frac{a_3}{a_7}\right|\right\}t \tag{3.1}$$

$$\left(\text{Since } \int_{-\infty}^{x} e^{\alpha(x-\eta)}d\eta = -\frac{1}{\alpha} > 0 \text{ for } \alpha<0 \text{ and } \int_{x}^{\infty} e^{\beta(x-\eta)}d\eta = \frac{1}{\beta} > 0, \beta>0\right).$$

Taking the supremum of both sides of (3.1) for $(x,t) \in \mathbb{R}\times[0,t_o]$ and using the definition of the norm in the space ℓ_{t_o}, we have

$$\left\|Au_1-Au_2\right\|_{\ell_{t_o}} \le \left\{\frac{1}{\sqrt{a_4^2-4a_7}}\left[2+\left|\frac{a_3}{a_7}\right|\sqrt{a_4^2-4a_7}+\frac{2|a_1|}{n+1}\left\|\sum_{i=0}^{n}u_1^{n-i}u_2^i\right\|_{\ell_{t_o}}\right]+\left|\frac{a_3}{a_7}\right|\right\}\left\|u_1-u_2\right\|_{\ell_{t_o}}$$

$$\le \left\{\frac{2}{\sqrt{a_4^2-4a_7}}\left[1+\frac{|a_1|}{n+1}\left\|\sum_{i=0}^{n}u_1^{n-i}u_2^i\right\|_{\ell_{t_o}}\right]+2\left|\frac{a_3}{a_7}\right|\right\}\left\|u_1-u_2\right\|_{\ell_{t_o}} \tag{3.2}$$

(from the definition of α and β), which implies that A is a continuous mapping of ℓ_{t_o} into itself.

Furthermore, it satisfies a Lipschitz condition with Lipschitz constant λ such that

$$\left\{\frac{2}{\sqrt{a_4^2-4a_7}}\left[1+|a_1|K^n\right]+2\left|\frac{a_3}{a_7}\right|\right\}t_o \le \lambda < 1, \tag{3.3}$$

where $\left\|\sum_{i=0}^{n}u_1^{n-i}u_2^i\right\|_{\ell_{t_o}} \le (n+1)K^n$.

Inserting the condition (3.3) into (3.2) yields

$$\left\|Au_1-Au_2\right\|_{\ell_{t_o}} \le t_o\left\{\frac{2}{\sqrt{a_4^2-4a_7}}\left[1+|a_1|K^n\right]+2\left|\frac{a_3}{a_7}\right|\right\}\left\|u_1-u_2\right\|_{\ell_{t_o}}$$

$$\le \lambda\left\|u_1-u_2\right\|_{\ell_{t_o}}. \tag{3.4}$$

Choosing $u_2=0$ and $u_1=u$ in (3.4), then

$$\left\|Au\right\|_{\ell_{t_o}} \le \lambda\left\|u\right\|_{\ell_{t_o}}, \quad 0<\lambda<1,$$

i.e., A is a contractive mapping for values of t_o, where

$$t_o\left\{\frac{2}{\sqrt{a_4^2-4a_7}}\left[1+|a_1|K^n\right]+2\left|\frac{a_3}{a_7}\right|\right\} \le \lambda < 1.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Similarly,
$$\left\|Bu\right\|_{\ell_{t_o}} \leq \lambda \left\|u\right\|_{\ell_{t_o}}. \tag{3.5}$$

But using (2.5) and (3.5), we have
$$\left\|Au\right\|_{\ell_{t_o}} \leq \sup_{x\in \mathbb{R}} |f(x)+1\left\|Bu\right\|_{\ell_{t_o}} \leq \sup_{x\in \mathbb{R}} |f(x)| + \lambda\left\|u\right\|_{\ell_{t_o}}.$$

Now, for the mapping A to be contractive for any t_o, we must have
$$\sup_{x\in \mathbb{R}} |f(x)| \leq (1-\lambda)K. \tag{3.6}$$

(It can be easily checked that (3.3) and (3.6) can be satisfied simultaneously by choosing $\lambda = \frac{1}{2}, K = 2M$ and $\sup_{x\in \mathbb{R}} |f(x)| < M$).

Hence A is contractive mapping for any value t_o.

Thus, according to the fixed point theorems for Banach spaces the integral equation (2.5) has a unique solution which is continuous and bounded for all t such that $0 \leq t \leq t_o$. This proves the following lemma.

LEMMA 3.1. *Let $f(x)$ be a continuous function such that $\sup_{x\in \mathbb{R}} |f(x)| \leq M < \infty$. Then, there exists a $t_o(M) > 0$ such that the integral equation (2.5) has a solution satisfying $u(x,0) = f(x)$ which is bounded and continuous for $(x,t) \in \mathbb{R} \times [0,t_o]$.*

Next, we turn to show that the solution guaranteed by Lemma 3.1 has sufficient regularity to be a classical solution of (1.1).

LEMMA 3.2. *If $f(x) \in C^2(\mathbb{R})$. Then any solution of (2.5) which is an element of ℓ_T (for given $T > 0$) is also an element of $l_T^{2,\infty}$.*

Proof. Let u be a solution of (2.5) guaranteed by Lemma 3.1, i.e.,

$$\begin{aligned} u = Au = f((x) &+ \frac{\alpha}{\sqrt{a_4^2 - 4a_7}} \int_o^t \int_{-\infty}^x e^{\alpha(x-\eta)} \left[(1+a_3\alpha)u + \frac{a_1}{n+1}u^{n+1}\right] d\eta d\tau \\ &+ \frac{\beta}{\sqrt{a_4^2 - 4a_7}} \int_o^t \int_x^\infty e^{\beta(x-\eta)} \left[(1+a_3\beta)u + \frac{a_1}{n+1}u^{n+1}\right] d\eta d\tau + \frac{a_3}{a_7}\int_o^t u d\tau. \end{aligned}$$

Since $u \in \ell_{t_o}$, then u is bounded and uniformly continuous on $\mathbb{R} \times [0,t_o]$. Hence $u = Au$ is continuously differentiable function, i.e., u_t exists and is given by

$$\begin{aligned} u_t = (Au)_t = &\frac{\alpha}{\sqrt{a_4^2 - 4a_7}} \int_{-\infty}^x e^{\alpha(x-\eta)} \left[(1+a_3\alpha)u + \frac{a_1}{n+1}u^{n+1}\right] d\eta \\ &+ \frac{\beta}{\sqrt{a_4^2 - 4a_7}} \int_x^\infty e^{\beta(x-\eta)} \left[(1+a_3\beta)u + \frac{a_1}{n+1}u^{n+1}\right] d\eta + \frac{a_3}{a_7}u. \end{aligned}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Furthermore, u_t is continuous and bounded on $\mathbb{R} \times [0, t_\circ]$. Hence u_{tt} exists and is given by

$$u_{tt} = \frac{\alpha}{\sqrt{a_4^2 - 4a_7}} \int_{-\infty}^{x} e^{\alpha(x-\eta)} \left[(1 + a_3\alpha)u_t + a_1 u^n u_t\right] d\eta$$
$$+ \frac{\beta}{\sqrt{a_4^2 - 4a_7}} \int_{x}^{\infty} e^{\beta(x-\eta)} \left[(1 + a_3\beta)u_t + a_1 u^n u_t\right] d\eta + \frac{a_3}{a_7} u_i.$$

Hence, by induction, the k-th derivatives with respect to t exists and is given by

$$\frac{\partial^k u}{\partial t^k} = \frac{\alpha}{\sqrt{a_4^2 - 4a_7}} \int_{-\infty}^{x} e^{\alpha(x-\eta)} \frac{\partial^{k-1}}{\partial t^{k-1}} \left[(1 + a_3\alpha)u + \frac{a_1}{n+1} u^{n+1}\right] d\eta$$
$$+ \frac{\beta}{\sqrt{a_4^2 - 4a_7}} \int_{x}^{\infty} e^{\beta(x-\eta)} \frac{\partial^{k-1}}{\partial t^{k-1}} \left[(1 + a_3\beta)u + \frac{a_1}{n+1} u^{n+1}\right] d\eta + \frac{a_3}{a_7} \frac{\partial^{k-1}}{\partial t^{k-1}} u, k = 2, 3, \ldots.$$

Since $u(x,t)$ is a solution of (2.5) in ℓ_T, then $u(x,0) = f(x)$ is continuous and bounded and u_x exists and is clearly given by

$$u_x = f'(x) + \frac{\alpha^2}{\sqrt{a_4^2 - 4a_7}} \int_{\circ}^{t} \int_{-\infty}^{x} e^{\alpha(x-\eta)} \left[(1 + a_3\alpha)u + \frac{a_1}{n+1} u^{n+1}\right] d\eta d\tau$$
$$+ \frac{\beta^2}{\sqrt{a_4^2 - 4a_7}} \int_{\circ}^{t} \int_{x}^{\infty} e^{\beta(x-\eta)} \left[(1 + a_3\beta)u + \frac{a_1}{n+1} u^{n+1}\right] d\eta d\tau$$
$$+ \int_{\circ}^{t} \left\{ \left(\frac{a_7 - a_3 a_4}{a_7^2}\right) u + \frac{a_1}{(n+1)a_7} u^{n+1} + \frac{a_3}{a_7} u_x \right\} d\tau.$$

This shows that u_x is continuous and bounded, then the first integral is a continuously differentiable function of x. Hence $f \in C^2(\mathbb{R})$, we can similarly differentiate again and obtain

$$u_{xx} = f''(x) + \frac{\alpha^3}{\sqrt{a_4^2 - 4a_7}} \int_{\circ}^{t} \int_{-\infty}^{x} e^{\alpha(x-\eta)} \left[(1 + a_3\alpha)u + \frac{a_1}{n+1} u^{n+1}\right] d\eta d\tau$$
$$+ \frac{\beta^3}{\sqrt{a_4^2 - 4a_7}} \int_{\circ}^{t} \int_{x}^{\infty} e^{\beta(x-n)} \left[(1 + a_3\beta)u + \frac{a_1}{n+1} u^{n+1}\right] d\eta d\tau$$
$$+ \int_{\circ}^{t} \left\{ \left[\frac{a_3 a_4^2 - a_3 a_7 - a_4 a_7}{a_7^3}\right] u - \frac{a_1 a_4}{(n+1)a_7^2} u^{n+1} \right.$$
$$\left. + \left[\frac{a_7 - a_3 a_4}{a_7^3}\right] u_x + \frac{a_1}{a_7} u^n u_x + \frac{a_3}{a_7} u_{xx} \right\} d\tau.$$

Thus, u_{xx} exists and is continuous and bounded.

Clearly u has the regularity of f in its x-derivatives and so $u \in l_T^{2,\infty}$. Which proves the lemma.

Combining Lemma 3.1 and Lemma 3.2, the proof of the following theorem is obtained.

THEOREM 3.1. *Let $f(x) \in C^2(\mathbb{R})$ and be bounded, then there exists a $t_\circ > 0$ such that the initial value problem* (1.1) *has classical solution for any $t, 0 \le t \le t_\circ$.*

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

THEOREM 4.1. *The solution of the initial value problem*

$$\begin{aligned} &u_t + u_x + a_1 u^n u_x + a_3 u_{xx} + a_4 u_{xt} + a_7 u_{xxt} = 0, \quad -\infty < x < \infty, \quad t \geq 0, \\ &u(x,0) = f(x), \end{aligned} \tag{1.1}$$

guaranteed by Theorem 3.1 is unique.

Proof. Let u and v be two solutions of the initial value problem (1.1) and $w = v - u$, then w statisfies the initial value problem

$$\begin{aligned} &w_t + w_x + \frac{a_1}{n+1}\left[w \sum_{i=0}^{n} v^{n-i} u^i\right]_x + a_3 w_{xx} + a_4 w_{xt} + a_7 w_{xxt} = 0, \\ &w(x,0) = 0. \end{aligned} \tag{4.1}$$

Multiplying (4.1) by w and integrating with respect to x between $x = -L$ and $x = L$, we have

$$\begin{aligned} 0 &= \int_{-L}^{L} \frac{1}{2}(w^2 - a_7 w_x^2)_t dx + \left[\frac{1}{2}w^2 + a_7 w w_{xt}\right]_{-L}^{L} \\ &+ \int_{-L}^{L} w\left[\frac{a_1}{n+1} w \sum_{i=0}^{n} v^{n-i} u^i + a_3 w_x + a_4 w_t\right]_x dx \\ &= \frac{1}{2}\int_{-L}^{L} \frac{d}{dt}(w^2 - a_7 w_x^2) dx + \left[\frac{1}{2}w^2 + a_7 w w_{xt} + a_3 w w_x + a_4 w w_t\right. \\ &+ \left.\frac{a_1}{n+1} w^2 \sum_{i=0}^{n} v^{n-i} u^i\right]_{-L}^{L} - \frac{a_1}{n+1}\int_{-L}^{L}\left(\sum_{i=0}^{n} v^{n-i} u^i\right) w w_x dx \\ &- \int_{-L}^{L} \left[a_3 w_x^2 + a_4 w_x w_t\right] dx. \end{aligned} \tag{4.2}$$

Since $u, u_x, u_t, u_{xt} v, v_x, v_t$ and v_{xt} vanish as $|x| \to \infty$, then so are w, w_x, w_t and w_{xt}. Hence as $L \to \infty$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}(w^2 - a_7 w_x^2) dx &= \frac{a_1}{n+1}\int_{-\infty}^{\infty} w w_x \left(\sum_{i=0}^{n} v^{n-i} u^i\right) dx + \int_{-\infty}^{\infty}\left[a_3 w_x^2 + a_4 w_x w_t\right] dx \\ &\leq \frac{|a_1|}{(n+1)} \sup_{x \in \mathbb{R}} \left|\sum_{i=0}^{n} v^{n-i} u^i\right| \int_{-\infty}^{\infty} w w_x dx + \int_{-\infty}^{\infty}\left[a_3 w_x^2 + a_4 w_x w_t\right] dx \\ &\leq \frac{|a_1|}{(n+1)\sqrt{-a_7}} C^n(t) \int_{-\infty}^{\infty} \frac{1}{2}(w^2 - a_7 w_x^2) dx + \int_{-\infty}^{\infty}\left[\left(\frac{2|a_3| + |a_4|}{2}\right) w_x^2 + \frac{|a_4|}{2} w_t^2\right] dx \end{aligned}$$

$$\left(\text{where } C^n(t) = \sup_{x \in \mathbb{R}} \left|\sum_{i=0}^{n} (v^{n-i} u^i)\right| \text{ and } a_7 < 0\right).$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Integrating with respect to t between 0 and t, yields

$$\int_{-\infty}^{\infty}(w^2 - a_7 w_x^2)dx \le \int_{-\infty}^{\infty}[w^2(x,0) - a_7 w_x^2(x,0)]dx \exp\left\{ \frac{|a_1|}{(n+1)\sqrt{-a_7}} \int_{\circ}^{t} C^n(\tau)d_\tau \right.$$

$$\left. + \int_{\circ}^{t} \left[\frac{\int_{-\infty}^{\infty}\left[\left(\frac{2|a_3| + |a_4|}{2}\right) w_x^2 + \frac{|a_4|}{2} w_\tau^2\right]dx}{\int_{-\infty}^{\infty} \frac{1}{2}(w^2 - a_7 w_x^2)dx} \right] d\tau \right\}$$

$$= 0, \text{ (Since } w(x,0) = 0), \tag{4.3}$$

i.e., $\int_{-\infty}^{\infty}(w^2 - a_7 w_x^2)dx = 0,$

which implies, clearly, that $w \equiv 0$, i.e., $u \equiv v$ for all $t > 0$ and Theorem 4.1 is proved.

Note that the choice of $f \in C^2(\mathbb{R})$ in the above analysis can be replaced by the assumption of $f \in H^2(\mathbb{R})$ (see Lemma 3.2). Hence Theorems 3.1 and 4.1 can be slightly changed to the following:

THEOREM 4.2. *Let $f \in H^m, m \ge 2$, then there exists a unique solution to the initial value problem* (1.1) *lies om $\mathcal{X}_T^m$ for all finite $0 \le T \le t_\circ$*

[For the definition of the function spaces H^m and $\mathcal{X}_T^m$ see Appendix B.]

5. Conclusion. In this paper we discussed the existance and uniqueness of the solution of a class of nonlinear dispersive dissipative differential equations and found that it has a unique solution through certain interval of time, i.e., locally. Since this class (1-1), as has been pointed out in the introduction, corresponds, equivalently, a subclass of the general class (1-4), then the existence and uniqueness guaranteed by Theorem 4- implies existence and uniqueness of the solution of the corresponding subclass. Unfortunately, we could not extend the solution globally. The reason behind this difficulty is thought to be the failure of finding more than one conservation law, which can be clearly obtained by rewriting (1-1) in a conserved form (El-Sherbiny, Hussein and Moustafa 1990).

APPENDIX A

THE GREEN'S FUNCTION OF (1.1)

Rewriting (1.1) in the form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{a_4}{a_7}\frac{\partial}{\partial x} + \frac{1}{a_7}\right)u_t = -\frac{1}{a_7}\frac{\partial}{\partial x}\left(u + \frac{a_1}{n+1}u^{n+1} + a_3 u_x\right) \tag{A.1}$$

it can be regarded as ordinary differential equations it u_t.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Following (Margenau and Murphy 1956), the general formula for the Green's function with the boundary conditions $u(x,t) \to 0$ as $|x| \to \infty$ is in the form:

$$G(x,\eta) = \begin{cases} \dfrac{z_1(\eta)z_2(x) - z_1(x)z_2(\eta)}{2p(\eta)\left[z_1(\eta)\dfrac{\partial}{\partial\eta}z_2(\eta) - z_2(\eta)\dfrac{\partial}{\partial\eta}z_1(\eta)\right] + Az_1(x) + Bz_2(x)}, & x \le \eta \\[2ex] -\dfrac{z_1(\eta)z_2(x) - z_1(x)z_2(\eta)}{2p(\eta)\left[z_1(\eta)\dfrac{\partial}{\partial\eta}z_2(\eta) - z_2(\eta)\dfrac{\partial}{\partial\eta}z_1(\eta)\right] + Az_1(x) + Bz_2(x)}, & x \ge \eta \end{cases} \tag{A.2}$$

where the constants A and B are found from the boundary conditions and $z_1(x) = e^{\alpha x}$, $z_2(x) = e^{\beta x}$,

$$\alpha = \frac{1}{2a_7}\left[-a_4 + \sqrt{a_4^2 - 4a_7}\right] < 0, \beta = \frac{1}{2a_7}\left[-a_4 - \sqrt{a_4^2 - 4a_7}\right] > 0$$

for all $a_4 \in (-\infty, \infty) - \{0\}, a_7 < 0$ and $p(\eta) = 1$.

Then, it follows that

$$G(x,\eta) = \begin{cases} \dfrac{e^{(\alpha\eta+\beta x)} - e^{(\alpha x+\beta\eta)}}{-2(\alpha-\beta)e^{(\alpha+\beta)\eta}} + Ae^{\alpha x} + Be^{\beta x}, & x \le \eta \\[2ex] \dfrac{e^{(\alpha\eta+\beta x)} - e^{(\alpha x+\beta\eta)}}{2(\alpha-\beta)e^{(\alpha+\beta)\eta}} + Ae^{\alpha x} + Be^{\beta x}, & x \ge \eta \end{cases}$$

$$= \begin{cases} \dfrac{a_7}{2\sqrt{a_4^2 - 4a_7}}\left[e^{\alpha(x-\eta)} - e^{\beta(x-\eta)}\right] + Ae^{\alpha x} + Be^{\beta x}, & x \le \eta \\[2ex] \dfrac{-a_7}{2\sqrt{a_4^2 - 4a_7}}\left[e^{\alpha(x-\eta)} - e^{\beta(x-\eta)}\right] + Ae^{\alpha x} + Be^{\beta x}, & x \ge \eta \end{cases} \tag{A.3}$$

and by using the boundary conditions as $|x| \to \infty$, we get

$$A = \frac{-a_7}{2\sqrt{a_4^2 - 4a_7}}e^{-\alpha\eta} \quad \text{and} \quad B = \frac{-a_7}{2\sqrt{a_4^2 - 4a_7}}e^{-\beta\eta}$$

(for all $\alpha < 0$ and $\beta > 0$).

Then (A.3) becomes

$$G(x,\eta) = \begin{cases} \dfrac{-a_7}{\sqrt{a_4^2 - 4a_7}}e^{\beta(x-\eta)}, & x \le \eta \\[2ex] \dfrac{-a_7}{\sqrt{a_4^2 - 4a_7}}e^{\alpha(x-\eta)}, & x \ge \eta \end{cases} \tag{A.4}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

which is the Green's function of the differential operator $\left(\frac{\partial^2}{\partial x^2} + \frac{a_4}{a_7}\frac{\partial}{\partial x} + \frac{1}{a_7}\right)$.

Appendix B

Definition Of Some Function Spaces

(1) $C^2(\mathbb{R}) = \left\{f(x); \int_{-\infty}^{\infty} |f^2| dx < \infty\right\}$ and $\left\|f\right\|^2_{C^2(\mathbb{R})} = \int_{-\infty}^{\infty} |f(x)|^2 dx.$

(2) $\ell_T = [C(\mathbb{R} \times [0,T])] = \{u(x,t); u$ is continuous and uniformly bounded on $\mathbb{R} \times [0,T]\}$ and

$$\left\|u\right\|_{\ell_T} = \sup_{\substack{x \in \mathbb{R} \\ 0 \le t \le T}} |u(x,t)|.$$

(3) $H^m(\mathbb{R})$ [Sobolev space of order m]

$$= \left\{f(x); f \in C^2(\mathbb{R}) \quad \text{and} \quad \frac{d^k f}{dx^k} \in C^2(\mathbb{R}), 1 \le k \le m\right\}$$

and

$$\left\|f(x)\right\|_{H^m(\mathbb{R})} = \sum_{k=0}^{m} \int_{-\infty}^{\infty} \left|\frac{d^k f}{dx^k}\right| dx = \sum_{k=0}^{m} \left\|\frac{d^k f}{dx^k}\right\|_{C^2(\mathbb{R})}$$

(4) $\mathcal{X}_T^m = C(0,T,H^m) = \{u(x,t); u : \mathbb{R} \times [0,T] \to \mathbb{R}$, for each $t \in [0,T], u(.,t) \in H^m$ and the mapping $u : [0,T] \to H^m$ is continuous and bounded$\}$ and $\left\|u\right\|^m_{\mathcal{X}_T} = \sup_{0 \le t \le T} \left\|u(.,t)\right\|_{H^m}$.

References

Benjamine, T. B., Bona, J. L. and **Mahony, J. J.** (1972) : Model equations for nonlinear dispersive systems, *Phil. Trans. Roy. Soc. London*, 272A, 47.

El-Sherbiny, H. M. (1987) : On the korteweg De Vries Equation and its Homologoues, *The City Univ.* London, PhD thesis.

El-Sherbiny, H. M., Hussein, S. A. and **Moustafa, K. A.** (1990) : Conservation laws of a class of differential equations (II, *Prof. Assiut Conf. of Maths. and stat.* (V), 71.

El-Sherbiny, H. M. (1992) : On a class of nonlinear dispersive dissipative differential equations—singular case—*Bull. Fac. Sci., Qena*, 4(1).

Margenau, H. and **Murphy, G. M.** (1956) : *The mathematics of Physics and Chemistry*, Affiliated East-West Press, Ltd. New Delhi (1956).

MATHEMATICS DEPARTMENT
FACULTY OF SCIENCE (QENA)
UNIVERSITY OF ASSIUT
QENA, EGYPT

AND

MATHEMATICS DEPARTMENT
FACULTY OF SCIENCE (ASSIUT)
UNIVERSITY OF ASSIUT
ASSIUT, EGYPT

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc., *86*, 491–496 (1994)

QUASI-HYPERBOLIC MANIFOLDS

N. H. ABDEL-ALL

(*Received 15 March 1991*)

Abstract. We investigate here the properties of a 10-dimensional manifold of linear elements (quasi-hyperbolic manifolds). Quasi-hyperbolic line manifolds of two dimensions (line congruences) as a submanifold of quasi-hyperbolic manifolds are classified, up to the rank of the attitude matrix, into four types. Existence theorems are given and finally, a construction of the Grassmann manifold, using Klein representation, is obtained.

Introduction. We consider a manifold generated by two real points $Q_\beta(\beta = 2, 3)$ belonging to two intersecting planes in the projective space P_3. We shall say that such a manifold is a manifold of linear elements and denote it by $K(3,1)$. The dimension of such manifold is equal to 10. In (Soliman and Abdel-All 1982), it was shown that in a partially canonical moving frame $\{A_i\}(i, j, k = 1, 2, 3, 4)$, the linear element (the absolutum) is determined by the following equations:

$$x^1 \pm x^4 = 0, \quad Q_\beta = Q_\beta(\delta_j^\beta) \tag{1}$$

where x^i are the homogeneous co-ordinates in the projective space P_3 and δ is the symbol of the well-known kroneker delta. The local co-ordinates $(u^n), (n = 1, 2, \ldots, 10)$ of $K(3,1)$ are furnished by the *1st* integrals of the completely integrable system $(DA_i = w_i^j A_j, D\omega_i^j = \omega_j^k \wedge \omega_k^j)$:

$$\omega_3^i = 0 \tag{2}$$

$$\omega_2^\gamma = 0, \quad (\gamma = 1, 3, 4) \tag{3}$$

$$\theta_\alpha = 0, \quad (\alpha = 1, 2, 3) \tag{4}$$

where

$$\theta_1 = \omega_4^1 - \omega_1^4, \theta_2 = \omega_4^4 - \omega_1^1, \theta_3 = \omega_2^2 + \omega_1^1 \tag{5}$$

and ω_i^j are the pfaff's forms.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Conditions (2) and (3) are in the form of differential equations for the stationary subgroup H_1^3 of the projective group $PG(3,R)$, while conditions (2), (3) and (4), taken together are differential equations for the stationary subgroup H_2^3 of $PG(3,R)$. The transformations in the subgroup H_1^3 do not move the two points Q_β, and the transformations in H_2^3 do not alter the linear element defined by (1) (Laptev 1953), (Pekarskiene 1977).

2. Non-Holonomic Manifolds. The manifold $K(3,1)$ can be viewed as a fibered space, a plane in P_3 being a point of the base (Grassmann manifold $Gr(2,3)$ in the dual space of P_3), and the fibers are spaces isomorphic to the 10-dimensional space of degenerate quadric surfaces (linear elements). The manifold $K(3,1)$ is called a quasi-hyperbolic manifold and the linear element (1) is considered as its absolutum (Abdurahmanova 1977), (Yasinskaya 1976).

Using a mapping of morphism type, we can distinguish a one-parameter submanifold $K(3,1;1)$ of $K(3,1)$ defined by the system of differential equations:

$$\omega_3^i = \lambda^i\theta, \;\; \omega_2^\gamma = \mu^\gamma\theta, \;\; \theta_\alpha = \nu_\alpha\theta, \;\; (D\theta = \theta\wedge\hat{\theta}) \tag{6}$$

and by the system of corresponding exterior quadratic equations. The system of quantities $H\Big(K(3,1;1)\Big) = \Big\{\lambda^i, \mu^\gamma, \nu_\alpha\Big\}$ forms the *1st* fundamental differential-geometric object of the one-parameter submanifold of the manifold of linear elements $K(3,1)$.

Since $\Delta\nu_\alpha = \nu_\alpha\widehat{\Pi}\{\Delta$ is the symbol for differentiation with respect to *2nd* order parameters, $\widehat{\Pi} = \widehat{\theta}(\Delta)\}$, then $h = \{\nu_\alpha\}$ is a subobject of the differential-geometric object H. The subobject h geometrically define the characteristic line of a one-parameter family of planes of degenerate *2nd* order curves (the fixed line (A_2, A_3)) in P_3.

In the projective group $PG(3,R)$ we can distinguish still another stationary subgroup H_3^3, the transformations of which do not move the linear element (1) and the line $\ell \equiv (A_1, A_4)$. This stationary subgroup is defined by the differential equations:

$$\omega_p^q = 0, \;\; \omega_3^i = 0, \;\; \omega_2^\gamma = 0, \;\; \theta_\alpha = 0, \;\; (p = 1,4; q = 2,3). \tag{7}$$

The system of differential equations:

$$\omega_1^q = a_\eta^q\omega_4^\eta + b_q^\alpha\theta_\alpha + C_i^q\omega_3^i + E_\gamma^q\omega_2^\gamma, \quad (\eta = 2,3) \tag{8}$$

and the corresponding exterior quadratic equations defines a non-holonomic manifold (Pekarskiene 1977). This is an involutive system and by using Cartan's methods of moving frames (Finikov 1948), (Wladyslaw 1970), we have proved the following:

THEOREM (1). *The non-uniqueness of the solutions of* (8) *being characterized by two functions in twelve arguments.*

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

3. Quasi-Hyperbolic Line Congruences. We consider a special case of (8) in which $b_q^\alpha = 0, C_i^q = 0, E_\gamma^q = 0$. We denote the resulting manifold by $Gr(1,3;2)$. The manifold $Gr(1,3;2)$ is defined by the system of linear differential equations:

$$\omega_1^\eta = a_q^\eta \omega_4^q, \tag{9}$$

of course, we suppose $\omega_4^2 \wedge \omega_4^3 \neq 0, \mathrm{Det}(a_q^\eta) \neq 0$) and the associated exterior quadratic equations. This is an involutive system, the nonuniqueness of its solution being characterized by two functions of two arguments. As is known, the equations (9) are the differential equations of a line congruence (two-dimensional submanifold in the domain of parameters u^n) having the line ℓ as a generating element.

From the stationarity conditions (4), we have obviously:

$$\Delta A_p = \Pi_p^\xi A_\xi, \quad (\xi = 1, 4), \tag{10}$$

where $\Pi_p^\xi = \omega_p^\xi(\Delta)$, and from (9) we see that the forms ω_i^q are principal forms. Hence we obtain the following differentials:

$$\left.\begin{aligned} DA_1 &\equiv a_q^\eta w_4^q A_\eta (\mathrm{mod}\ A_1, A_4) \\ DA_4 &\equiv w_4^q A_q (\mathrm{mod}\ A_1, A_4) \end{aligned}\right\}. \tag{11}$$

The space $(A_1, A_4, a_2^q, A_q, a_3^q A_q)$ is called the tangent space $T = T(\omega_4^q)$ of the line congruence $Gr(1,3;2)$ at the line $\ell = \ell(\omega_4^q)$. Evidently, $1 \leq \dim T \leq 3$. The number $\dim T$ is called the character of the line congruence $Gr(1,3;2)$. Thus, we have the following:

LEMMA (1). *The line congruences* (9) *having the character three.*

Expanding the exterior quadratic equations corresponding to (9) *by Cartan's lemma, we get that the equations for the infinitesimal variations of quantities a_q^η, when the 1st order parameters are fixed, are $\Delta a_q^\eta = \theta_{qp}^\eta \Pi_1^p$ where*

$$\left.\begin{aligned} &\theta_{\eta 1}^\eta = 0, \theta_{31}^2 = 2a_2^3, \theta_{21}^3 = -2a_3^2, \\ &\theta_{24}^2 = 1 - \left(a_2^2\right)^2 - a_3^2 a_2^3, \theta_{34}^2 = -\left(a_2^2 + a_3^3\right) a_2^3, \\ &\theta_{24}^3 = -\left(a_2^2 + a_3^3\right) a_3^2, \theta_{34}^3 = \left(a_3^3\right)^2 + a_2^3 a_3^2 - 1, \end{aligned}\right\}. \tag{12}$$

4. Classification of Line Congruences. In general, the matrix $M = (\theta_{qp}^\eta)$ has a rank equal two. We give therein a complete classification of line congruences (9), that is based on the rank of the attitude matrix M (Soliman and Abdel-All 1982), for which M has a rank lower than two. Therefore, we have the following types of line congruences:

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Type I, for which the rank of the matrix M equal zero, is given by the complete integrable system of differential equations:

$$\omega_1^2 \mp \omega_4^2 = 0, \omega_1^3 \pm \omega_4^3 = 0 \tag{13}$$

These line congruences are studied in (Soliman and Abdel-All 1982) as submanifolds of line complexes (3-dimensional line manifolds). Type II, for which the rank of the matrix M equal one, splitted into three subtypes $\mathrm{II}_\alpha(\alpha = 1, 2, 3)$ which are given by the following differential equations:

$$\left.\begin{aligned} &\omega_1^2 - a_2^2\omega_4^2 = 0, \omega_1^3 - a_3^3\omega_4^3 = 0, (a_2^2 - a_3^3 \neq 0) \\ &Da_2^2 = \left(1 - (a_2^2)^2\right)\omega_1^4 + \lambda_2\omega_4^2, \\ &Da_3^3 = \left(1 - (a_3^3)^2\right)\omega_1^4 + \lambda_3\omega_4^3, \end{aligned}\right\} \tag{14$_1$}$$

$$\left.\begin{aligned} &\omega_1^2 - a_2^2\omega_4^2 = 0, \omega_1^3 \mp \omega_4^3 - a_2^3\omega_4^2 = 0 \\ &Da_2^2 = \left(1 - (a_2^2)^2\right)\omega_1^4 + \mu_2\omega_4^2, (a_2^2 \neq \pm 1) \\ &Da_2^3 = a_2^3(a_2^2 \mp 1)\omega_1^4 - 2a_2^3\omega_1^1 + \rho_2\omega_4^2 \end{aligned}\right\} \tag{14$_2$}$$

and

$$\left.\begin{aligned} &\omega_1^3 - a_3^3\omega_4^3 = 0, \omega_1^2 \mp \omega_4^2 + a_3^2\omega_4^3 = 0 \\ &Da_3^3 = \left(1 - (a_3^3)^2\right)\omega_1^4 + \mu_3\omega_4^3, (a_3^3 \neq \pm 1) \\ &Da_3^2 = a_3^2(a_3^3 \mp 1)\omega_1^4 + 2a_3^2\omega_1^1 + \rho_3\omega_4^3 \end{aligned}\right\} \tag{14$_3$}$$

respectively, where λ_q, μ_q, ρ_q are invariants of the 2nd order contact elements of the generating element 2 of line manifold. Using Cartan's common methods for the systems (13), $(14)_1, (14)_2, (14)_3$ we have proved the following existence theorems:

THEOREM (2). *The line congruences of type I exist within two arbitrary constants.*

THEOREM (3). *The range of existence of line congruences of types II_α comprises with two arbitrary functions of one argument.*

5. Developables of Line Congruences. To choose a ruled surface in Gr. (1, 3; 2), means to choose functions $\omega_4^q = \omega_4^q(\phi)$ in the parameter $\phi(d\phi = 0)$. The developable surfaces of Gr. (1,3; 2) are given by the quadratic differential equation

$$a_3^2(\omega_4^3)^2 + (a_2^2 - a_3^3)\omega_4^2\omega_4^3 - a_2^3(\omega_4^2)^2 = 0 \tag{15}$$

Generally, Gr (1, 3; 2) is non-parabolic with two quasi-focal surfaces (F) where $F = A_1 + tA_4$ is a quasi-focal point and t is a root of the quadratic equation

$$t^2 + tr(a_q^\eta)t + Det(a_q^\eta) = 0 \tag{16}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$(a_2^2 - a_3^3)^2 - 4a_2^3 a_3^2 > 0$. Hence Gr (1, 3; 2) have two layers of quasi-developable surfaces. developables of line congruences (13) are studied in (Soliman and Abdel-All 1982).

The two layers of developables of line congruences of types $\mathrm{II}_1; \mathrm{II}_2$ and II_3 are denoted ctively by $\mathrm{II}_{11}, \mathrm{II}_{12}; \mathrm{II}_{21}, \mathrm{II}_{22}$ and $\mathrm{II}_{31}, \mathrm{II}_{32}$ and are given by the following systems of linear rential equations:

$$\omega_4^2 = 0, \omega_1^2 = 0, \omega_1^3 - a_3^3\omega_4^3 = 0, \tag{II_{11}}$$

$$\omega_4^3 = 0, \omega_1^2 - a_2^2\omega_4^2 = 0, \omega_1^3 = 0, \tag{II_{12}}$$

$$\omega_4^2 = 0, \omega_1^2 = 0, \omega_1^3 \mp \omega_4^3 = 0, \tag{II_{21}}$$

$$(a_2^2 \pm 1)\omega_4^3 - a_2^3\omega_4^2 = 0, \omega_1^2 - (a_2^2/a_2^3)(a_2^2 \pm 1)\omega_4^3 = 0;$$

$$\omega_1^3 - a_2^2\omega_4^3 = 0, \tag{II_{22}}$$

$$\omega_4^3 = 0, \omega_1^3 = 0, \omega_1^2 \mp \omega_4^2 = 0, \tag{II_{31}}$$

and $$a_3^2\omega_4^3 - (a_3^3 \mp 1)\omega_4^3 = 0, \omega_1^3 - a_3^3\omega_4^3 = 0,$$

$$\omega_1^2 + a_3^2(a_3^3 \mp 2)\omega_4^3 = 0, \tag{II_{32}}$$

pectively. Thus, we have proved the following:

LEMMA (2). *The two layers of developables of line congruences II_α are characterized the involutive systems of differential equations $(II_{\alpha\xi}), \xi = 1, 2$.*

6. General Quasi-Hyperbolic Line Manifolds. Let us use the well-known representation of lines of the projective 3-space P_3 onto the points of Klein quadric (K-quadric) the projective 5-space (K-space) P_5^k (Abdel-All 1989). The associated frame in P_5^k is ven by $A_{ij} (i \neq j)$, where $A_{ij} = [A_i, A_j]$ denotes the Grassmann products of the two points $_i, A_j \in \{A_i\}$. The fundamental equations of the frame $\{A_{ij}\}$, up to the stationarity conditions the subgroup H_2^3, are given by the following:

$$DA_{ij} \equiv w_i^k A_{kj} + \omega_j^k A_{ik} \pmod{A_{ij}} \tag{17}$$

From these differentials, it is easy to see that the space P_5^k contains hyperspace $_4 \equiv (A_{12}, A_{13}, A_{23}, A_{24}, A_{34})$, on it there exist 2-dimensional quasi-hyperbolic planes $\Sigma_2' = A_{12}, A_{23}, A_{24}), \Sigma_4'' = (A_{13}, A_{23}, A_{34})$ and quasi-hyperbolic point $A_{23} \in \Sigma_2' \cap \Sigma_2'' \subset \Sigma_4$. The configuration $\Sigma \equiv \{A_{23}, \Sigma_2', \Sigma_2'', \Sigma_4\}$, which invariant under the transformations of the subgroup H_2^3, is called the absolutum of the klein space P_5^k.

DEFINITION. *The invariant points Q_β are called quasi-hyperbolic points and every manifold contains one of these points is called quasi-hyperbolic manifold.*

From (17) and the invariant conditions of the subgroup H_2^3 for the configuration Σ (the absolutum of P_5^k). We have proved the following:

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

THEOREM. *The Grassmann manifold Gr.* $(1,3)$ *in the quasi-hyperbolic space is considered as the union of quasi-hyperbolic line-manifolds of dimension* 2 *and non-quasi-hyperbolic line manifolds of dimenion* 4.

References

Abdurahmanova, H. K. (1977) : Surfaces in quasi-hyperbolic space. *Izv. Akad, Nauk. USSR. Ser. Fiz. Mat. Nauk* 2, 3,

Abdel-All, N. H. (1989) : Klein correspondences of a class of line complexes; *Tensor, N. S.*, 48, 108.

Finikov, S. P. (1948) : *Cartan's methods of exterior forms and their applications in differential geometery*, Moscow.

Laptev, G. F. (1953) : Differential geometry of immersed manifolds. Group-theoretic method for differential geometric investigations. *Proc. Moscow, Math. Soc.* 2, *scow*, 275.

Pekarskiene, A. Yu. (1977) : Some problems in the geometry of a semiholonomic manifold of quadratic elements. *Liet. Math. Rinkinys*, 17, 4, 177.

Soliman, M. A. and **Abdel-All, N. H.** (1982) : A classification of line complex immersed in quasi-hyperbolic space that based on the mobility of a repere. *Indian J. pure appl. Math.*, 13 (3) : 323.

Wladyslaw, S. (1970) : *Exterior forms and their applications.* Warsaw.

Yasinskaya, E.U. (1976) : Metric invariants and covariants of linear complexes in real non-Euclidean and Semi non-Euclidean spaces. *Ukrainskii Math. Zhurnal*, 28, 2, 271.

MATHEMATICS DEPARTMENT
FACULTY OF SCIENCES
ASSIUT UNIVERSITY
ASSIUT, EGYPT

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Cal. Math. Soc., 86, 497–502 (1994)

SMALL M-PROJECTIVE MODULES AND SUBMODULES OF PSEUDO-PROJECTIVE MODULE

B. M. PANDEYA

(*Received 1 November, 1991*)

Abstract. Small M-projective modules are introduced here. We give characteri-on of such modules and study some of their properties. We provide characterization of s over which each submodule of a finitely generated pseudo-projective module is pseudo-jective.

1. Introduction. In section three of this paper we have generalized several results (Azumaya) for small M-projective modules. It is well known that over a hereditary ring each module of a projective module is projective but it is not true in the case of quasi-projective pseudo-projective modules. Rings for which each submodule of a quasi-projective module is si-projective were first studied by Fuller (1972). He called these rings completely hereditary s and proved that an artinian ring is a completely hereditary ring if, and only if, every ple module is projective or injective. Later on J. Ahsan (1991) has studied those rings which each submodule of a finitely generated quasi-projective module is quasi-projective. has provided characterizations of hereditary and von Neumann regular rings by using the perties of M-projective modules. As an application of results proved for small M-projective dules we have proved that among Noetherian rings, the rings over which each submodule of initely generated pseudo-projective module is pseudo-projective, are precisely those whose factor rings are hereditary. This generalizes the result of J. Ahsan.

2. Preliminaries. Throughout this paper we shall assume that each ring is asso-tive with identity element. Every R-module is left R-module. An R-module P is said to be projective if, given an epimorphism π of M onto another module N, every homomorphism: $\to N$ can be lifted to a homomorphism: $P \to M$ relative to π. An R-module P is called small projective if given a minimal epimorphism π: $M \to N$ and an epimorphism $g : P \to N$, re exists a homomorphism $h : P \to M$ such that $g = \pi o h$. An R-module M is called pseudo-jective (Tiwary and Pandeya 1979) if given a natural epimorphism $\pi : M \to N$ and an morphism $g : M \to N$, there exists a homomorphism $h : M \to M$ such that $g = \pi o h$. Basic perties of pseudo-projective modules are given in (Tiwary and Pandeya 1979) and those M-projective modules in (Anderson and Fuller 1973) and (Azumaya). A ring R is called

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

von Neumann regular if for each $r \in R$ there exists $x \in R$ such that $r = r \times r$. If A is a sub of an R-module M then ann (A) will denote the annihilator of A in R. For an R-module $M^{(A)}$ will denote the direct sum of A copies of M and $M^{(n)}$ will mean the direct sum of copies of M, where $n > 1$. An R-module M is faithful if and only if R can be embedded $M^{(A)}$ for some set A. An R-module M is called cofaithful if R can be embedded in a fi direct sum of its copies.

3. Small M-Projective Modules. We begin with the following lemmas:

LEMMA 3.1. *If a direct sum of R-module is small M-projective, then each summand is sm M-projective.*

Proof is straightforward.

LEMMA 3.2. *Let N be an R-module, which is either a submodule of M or a homomorp image of M under minimal epimorphism. Then every small M-projective module is sm N-projective.*

Proof is analogous to the proof of Proposition 3 (Azumaya).

DEFINITION. *Let A, B be left R-modules and $\phi : A \to B$ be a minimal epimorphism. W shall call ϕ a minimal M-epimorphism if there exists an epimorphism $g : A \to M$ such th* $\ker \phi \cap \ker g = 0$.

THEOREM 3.3. *Let M and P be left R-modules. Then the following conditions are equivale*

(i) *P is small M-projective,*

(ii) *Given any minimal M-epimorphism $\phi : A \to B$ and any epimorphism $h : p \to B$, there ex a homomorphism $f : P \to A$ such that $h = \phi$ of,*

(iii) *Every minimal M-epimorphism onto P splits.*

Proof. (i) $\to$ (ii) since ϕ is a minimal M-epimorphism, there exists an epimorphism $g : A \to$ such that $\ker \phi \cap \ker g = 0$. Let $\pi : M \to N$ be the natural epimorphism, where $N = M/g(\ker$ Since $\ker \phi$ is small in A, therefore, $g(\ker \phi)$ is small in M. Hence π is a minimal epimorphi By homomorphism decomposition theorem there exists a homomorphism $\mu : B \to N$ such th $\mu o \phi = \pi o g$.

Since $\mu o \phi$ is onto, therefore, μ is onto. Hence $\mu o h$ is an epimorphism. By (i) th exists a homomorphism $k : P \to M$ such that $\pi o k = \mu o h$.

Since ϕ is onto, for every $x \in P$, there exists an $a \in A$ such that $\phi(a) = h(x)$, then w have $\pi o g(a) = \mu o \phi(a) = \mu o h(x) = \pi o k(x)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

This implies that $g(a)-k(x) \in \ker \pi$. Therefore, $g(a)-k(x) = g(a_\circ)$ for some $a_\circ \in \ker \phi$. Put $a' = a - a_\circ$ then $\phi(a') = \phi(a) - \phi(a_\circ) = \phi(a) = h(x)$ and $g(a') = g(a) - g(a_\circ) = k(x)$. Thus, a' satisfies two equations $\phi(a') = h(x)$ and $g(a') = k(x)$. But since $\ker \phi \cap \ker g = 0$ implies a' is the only element of A satisfying both the equations for a given x. Thus, we have a homomorphism $f : P \to A$ such that ϕ of $(a) = \phi(a') = h(x)$ for each $x \in P$, i.e., ϕ of $= h$

(ii) $\Rightarrow$ (iii), (iii) is a special case of (ii).

(iii) $\Rightarrow$ (i) Let $\psi : M \to N$ be the minimal epimorphism and $h : P \to N$ any epimorphism. Consider the direct sum $M \oplus P$. Define $g : M \oplus P \to N$ by $g(u,x) = \psi(u) - h(x)$ which is indeed an epimorphism. Let $K = \ker g$. Define $\mu : K \to M$ by $(\mu(u,x) = u, \pi : K \to P$ by $\pi(u,x) = x$, then π and μ are epimorphisms, for let $x \in P$, then there exists $u \in M$ such that $\psi(u) = h(x)$ or $\psi(u) - h(x) = 0$. This implies that $(u,x) \in \ker g$ so $x = \pi(u,x)$ is in the image of $\pi(K)$. Similar arguments can be applied to μ. Now $\ker \pi \cap \ker \mu = 0$. It can be easily seen that $\ker \pi \subset \ker \psi \subset \ker g$.

Claim : $\ker \pi$ is small in k. Let $K' + \ker \pi = K$, then $K' + \ker \psi = K$ because $\ker \pi \subset \ker \psi$.

But $\ker \psi$ is small in M and is contained in K. Therefore, $\ker \psi$ is small in K (by lemma, 1 (Leonard 1966)).

Hence $K = K'$.

Therefore, $\ker \pi$ is small in K. This implies that π is minimal epimorphism. Thus, π is a minimal M-epimorphism, hence it splits by (iii), i.e., there exists a homomorphism $\eta : P \to K$ such that $\pi o \eta = 1_P$.

On the other hand $\psi o \mu = h o \pi$, since for any $P \in K, P = (u,x)$, we have

$$\psi o \mu(u,x) - h o \pi(u,x) = \psi(u) - h(x) = g(u,x) = 0,$$

since $(u,x) \in K$. Put $f = \mu o \eta$, then $\psi o f = \psi o \mu o \eta = h o \pi o \eta = h i_P = h$. Hence P is small M-projective.

COROLLARY 3.4. (Azumaya Theorem 4) *Let M and Q are R-modules then the following conditions are equivalent:*

(i) *Q is M - projective;*

(ii) *Given any M-epimorphism $\psi : A \to B$ and any homomorphism*

$g : Q \to B$, there exists a homomorphism;

$f : Q \to A$, such that ψ of $= g$;

(iii) *Every M-epimorphism onto Q splits.*

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

THEOREM 3.5. *Let M_1 and M_2 be left R-modules and let P be small M_1 and small M_2-projective. Then P is small $M_1 \oplus M_2$-projective. The proof is on the same lines as given by* (Azumaya Theorem 5).

PROPOSITION 3.6. *Every finitely generated small R-projective module is projective R-module.*

Proof. Let P be finitely generated and small R-projective module, then the small M-epimorphism from $R^{(n)}$ onto P splits because P is small R-projective and hence small $R^{(n)}$-projective by Theorem 3.5. Therefore, P is isomorphic to a direct summand of $R^{(n)}$, hence P is projective.

PROPOSITION 3.7. *Let M be a cofaithful module. Then every finitely generated small M-projective module is projective.*

Proof. If M is cofaithful then R can be embedded in $M^{(n)}$ for some n. If P is finitely generated small M-projective then P is small $M^{(n)}$-projective by Theorem 3.5 and, therefore, small R-projective by lemma 3.2 then P is projective by Proposition 3.6.

COROLLARY 3.8. *Every finitely generated cofaithful pseudo-projective module is projective.*

COROLLARY 3.9. *Every finitely generated cofaithful small-projective module is projective.*

COROLLARY 3.10. *Let M be a cofaithful R-module. Then every finitely generated M-projective R-module is projective.*

PROPOSITION 3.11. *Let P be a finitely generated R-module and let $\{M_\alpha/\alpha \in I\}$ be a family of R-modules such that P is small M_α-projective for each $\alpha \in I$. Then P is small $M = \oplus\Sigma M_\alpha$-projective.*

Proof is similar to the proof of Proposition (Azumaya).

COROLLARY 3.12. (Azumaya) *Let P be a finitely generated R-module and let $\{M_\alpha/\alpha \in I\}$ be a family of R-modules such that P is M_α-projective for each $\alpha \in I$, then P is $M = \oplus\Sigma M_\alpha$-projective.*

PROPOSITION 3.13. *Let P, Q and M be left R-modules and $\pi : P \to$ be minimal epimorphism. If Q is small M-projective then $h(\ker \pi) = 0$ for all epimorphism $h : P \to M$ and P is small M-projective.*

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. Let $h : P \to M$ be an epimorphism. Put $\ker\pi \cap \ker h = L$, and consider the factor module P/L. There can be defined epimorphisms $\delta : P \to P/L, \pi_1 : P/L \to Q$ and $h_1 : P/L \to M$ in the natural manner so that they satisfy $\ker\delta = L$.

$\ker\pi_1 = \ker\pi/L, \ker h_1 = \ker h/L$ and $\pi_1 o\delta = \pi$ and $h_1 o\delta h$. It can be shown that π_1, is minimal epimorphism. Then $\ker\pi_1 \cap \ker h_1 = (\ker\pi \cap \ker h)/L = 0$ which shows that π_1, is a minimal M-epimorphism. Since Q is small M-projective, therefore, it follows from Theorem 3.3 that π_1 splits, that is, there exists a monomorphism $\epsilon : Q \to P/L$ such that $\pi_1 o\epsilon = I$ on Q or $P/L = \ker\pi \oplus \epsilon(Q)$. But $\pi_1 o\delta = \pi$ is a minimal epimorphism. Hence $\epsilon(Q) = P/L$. Thus, $\ker\pi/L = \ker\pi_1 = 0$ or $\ker\pi = L$ which means that $\ker\pi \subset kerh$. In order to prove that P is small M-projective, consider any minimal epimorphism $\phi : M \to N$ and epimorphism $g' : P \to N$. Since N is homomorphic image of M under a minimal epimorphism, Q is N-projective. Hence we have $g'(\ker\pi) = 0$ as it has been shown above. But this implies that there can be defined a homomorphism $g : Q \to N$ such that $go\pi = g'$. Since Q is small M-projective, there exists a homomorphism $f' : P \to M$ which satisfies $\phi of = \phi ofo\pi = go\pi = g'$.

Thus, P is small M-projective.

As an application of above results we prove the following theorem:

THEOREM 3.14. *Let R be a Noetherian ring. Then the following conditions are equivalent:*

(i) *Each submodule of a finitely generated pseudo-projective R-module is pseudo-projective.*

(ii) *Each factor ring of R is hereditary.*

In the light of our Prop. 3.6, 3.13, 3.8 and lemma 3.2 the proof of the above theorem can be constructed on the lines of J. Ahsan (1991).

COROLLARY 3.15. (Ahsan 1991) *Let R be a Noetherian ring. Then the following conditions are equivalent:*

(i) *Each submodule of a finitely generated quasi-projective module is quasi-projective.*

(ii) *Each factor ring of R is hereditary.*

THEOREM 3.16. *Let R be a commutative ring. Then R is von Neumann regular ring if, and only if, each finitely generated submodule of a finitely generated pseudo-projective module is pseudo-projective.*

With the help of our Theorem 3.14 we have the above result on the lines of J. Ahsan (1991).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

COROLLARY 3.17. *Let R be a commutative ring. Then R is von Neumann regular ring if, and only if, each finitely generated submodule of a finitely generated quasi-projective module is quasi-projective.*

References

Anderson, F. W. and **Fuller, K. R.** (1973) : *Rings and categories of modules.* Springer Verlag, New York.

Azumaya, G. (1991) : –M projective and *M*-injective modules (*unpublished*).

Ahsan, J. (1991) : Submodules of a quasi-projective modules comm. in Algebra 9(9), 975.

Bass, H. (1960) : Finistic dimension and homological generalization of semi prime rings *Trans. Amer. Math. Soc.*, 95, 466.

Faith, C. (1973) : Algebra *Rings; Modules; Categories, II,* Springer Verleg New York.

Faith, C. (1976) : *Algbera II; Ring theory,* Springer Verlog. New York.

Fuller, K. R. (1969) : On Direct representation of Quasi-projectives and Quasi-injective, *Arch. Math.*, *20*, 495.

Fuller, K. R. (1970) : Correction to direct representation of Quasi-projective; *Arch. Math. 21*, 478.

Fuller K. R. (1972) : Relative projective and injectivity classes determined by simple modules: *J. London Math. Soc.*, *5*, 423.

Golan, J. S. (1970) : Characterisation of rings using Quasi-projective modules; *Israel J. Math.*, *8*, 34.

Golan, J. S. (1972) : Quasi perfect modules. *Quart J. Math.*, *41*, *3*, 687.

Leonard, W. W. (1966) : Small Modules, *Proc. Amer. Math. Soc. 19*, 2, 527.

Morita, K. (1966) : On the *S*-rings in the sense of F. Kasch, *Nayoga Math. J. 27*, 687.

Tiwari, A. K. and **Chaubey, K. N.** (1985) : Small projective modules. *Indian J. of pure appl. Math.*, *16(2)*, 133.

Tiwary, A. K. and **Pandeya, B. M.** (1979) : Pseudo-projective and pseudo-injective modules, O. *Indian J. pure and appl. Math. 9, 9*, 941.

DEPARTMENT OF APPLIED MATHEMATICS
INSTITUTE OF TECHNOLOGY
BANARAS HINDU UNIVERSITY
VARANASI 221005 (INDIA)

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc., *86*, 503–508 (1994)

ON CONTINUITY OF MAPPINGS IN D-METRIC SPACES

B. C. DHAGE

(Received 10 December 1991; revision received 16 December 1992; 2nd revision received 19 April 1994)

Abstract. In this paper some results in D-metric spaces are obtained. The continuity of the mappings in D-metric spaces is discussed via $(\epsilon - \delta)$-condition and the convergence of the sequence has been considered. The relation between the continuity theorems in ordinary and D-metric spaces is also discussed.

1. Introduction. Recently the present author (Dhage 1984, 1992) has introduced the notion of a D-metric space which is as follows.

DEFINITION 1.1. *Let X be a non-empty set. A real function D on $X \times X \times X$ is said to be a D-metric on X if it satisfies the following properties:*

(i) $D(x, y, z) \geq 0$ *for all* $x, y, z \in X$ *and equality holds if, and only if,* $x = y = z$, *(nonnegativity)*

(ii) $D(x, y, z) = D(x, z, y) = \ldots$ *(symmetry)*

(iii) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$ *for* $x, y, z, a \in X$. *(tetrahedral inequality)*

A non-empty set X together with a D-metric D is called a D-metric space and it is denoted by (X, D). A D-metric D is also called the diameter function of the points of X. The generalizations of a D-metric space with D-metric as a function of n variables are given in Dhage (1984).

EXAMPLE 1.1. Let $\mathbb{R}$ denote the real line. Then the function $D : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$D(x, y, z) = k \max\{\|x - y\|, \|y - z\|, \|z - x\|\} \tag{1.1}$$

for $x, y, z \in \mathbb{R}^n, k > 0$, where $\|.\|$ denotes a norm in the Euclidean space $\mathbb{R}^n$, is a D-metric on $\mathbb{R}^n$. Hence $(\mathbb{R}^n, D)$ is a D-metric space.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Let (X, D) be a D-metric space. Let $x_0 \in X$ be fixed and $\epsilon > 0$ given. We consider the sets $B^*(x_0, \epsilon)$ and $B(x_0, \epsilon)$ in X defined by

$$B^*(x_0, \epsilon) = \{y \in X : D(x_0, y, y) < \epsilon\}$$

$$\text{and} \quad B(x_0, \epsilon) = \{y \in X : D(x_0, y, z) < \epsilon\} \tag{1.2}$$

$$= \{y \in B^*(x_0, \epsilon) : \text{ if } y, z \in B(x_0, \epsilon), \text{ then } D(x_0, y, z) < \epsilon\}. \tag{1.3}$$

It is shown in Dhage (1993) that the set $B^*(x_0, \epsilon)$ and $B(x_0, \epsilon)$ are open balls if the D-metric D satisfies the condition viz.,

$$\text{(iv)} \quad D(x, y, y) \leq D(x, z, z) + D(z, y, y) \tag{1.4}$$

for $x, y, z \in X$.

Further it is shown in Dhage (1994) that the collections $\{B^*(x, \epsilon) : x \in X\}$ and $\{B(x, \epsilon) : x \in X\}$ induce the topologies on X which we denote by τ^* and τ, respectively. In this paper we implicitly assume that the D-metric D satisfies the condition (iv) and deal with the topology τ on a D-metric space X generated by the D-metric D on X. It is proved that (see for example Dhage (1993)) the D-metric D is a continuous function on $X \times X \times X$ w.r.t. the topology τ on X.

2. Continuous Mappings. In this section we discuss the continuity of mappings in a D-metric space with respect to the topology τ generated by the D-metric D in it. Before stating the main results of this paper we give some useful terminologies.

A topological space X is said to be D-metrizable if there exists a D-metric D on X that induces the topology of X. A D-metric space X is D-metrizable space together with the specific D-metric D that induces the topology of X.

A set U is open in the topology τ in X generated by the D-metric D if and only if for each $x \in U$, there exists a $\delta > 0$ such that $B(x, \delta) \subset U$. Similarly a set V in (X, D) is called closed if its complement X/V is open in X.

DEFINITION 2.1. *A mapping $f : (X, D_X) \to (Y, D_Y)$ is said to be continuous if U is any open set in Y, then $f^{-1}(U)$ is open in X.*

THEOREM 2.1. *Let X and Y be two D-metrizable spaces with D-metrics D_X and D_Y respectively. Then the mapping $f : X \to Y$ is continuous if and only if, for $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$D_X(x, y, z) < \delta \Rightarrow D_Y(fx, fy, fz) < \epsilon. \tag{2.1}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. Suppose that $f : X \to Y$ is continuous. Let $\epsilon > 0$ be given and consider the set $B_{D_Y}(fx, \epsilon)$ which is open in Y. Then the set $f^{-1}(B_{D_Y}(fx, \epsilon))$ is open in X containing $x \in X$. So, it contains some δ-ball $B_{D_X}(x, \delta)$ centered at x. If y and z are in $B_{D_X}(x, \delta)$, then fy and fz are in $B_{D_Y}(fx, \epsilon)$. By definition of ball which implies that $D_X(x, y, z) < \delta$ implies $D_Y(fx, fy, fz) < \epsilon$.

Conversely, suppose that for a given $\epsilon > 0$ there exists $\delta > 0$ such that $D_X(x, y, z) < \delta \Rightarrow D_Y(fx, fy, fz) < \epsilon$. Let U be an open set in Y containing the ball

$$B_{D_Y}(fx, \epsilon) = \{fy, fz \in Y : D_Y(fx, fy, fz) < \epsilon\},$$

centered at fx of radius ϵ. Then it suffices to show that $f^{-1}(U)$ is open in X. Let $x \in f^{-1}(U)$ then $fx \in U$. By $(\epsilon - \delta)$ condition, there is a ball

$$B_{D_X}(x, \delta) = \{y, z \in X : D(x, y, z) < \delta\}.$$

centered at x of radius δ such that $f(B_{D_X}(x, \delta)) \subset B_{D_Y}(fx, \epsilon)$. Therefore, $B_{D_X}(x, \delta)$ is a neighbourhood of x contained in $f^{-1}(U)$. Hence $f^{-1}(U)$ is open in X. This completes the proof.

Under certain situation the above Theorem 2.1 can be related to the well-known result concerning the continuity of mappings in ordinary metric spaces. Let the D-metric D_X and D_Y on X and Y be given by

$$D_X(x, y, z) = \max\{d_X(x, y), d_X(y, z), d_X(z, x)\} \tag{2.2}$$

$$\text{and} \quad D_Y(x, y, z) = \max\{d_Y(x, y), d_Y(y, z), d_Y(z, x)\} \tag{2.3}$$

where d_X and d_Y are ordinary metrics on X and Y respectively.

COROLLARY 2.1. *Let X and Y be two D-metrizable spaces with D-metrics D_X and D_Y given by (2.2) and (2.3), respectively. Then the continuity of the mapping $f : X \to Y$ w.r.t. ordinary metrics implies the continuity of f w.r.t. D-metrics and vice versa.*

Proof. Let X and Y be two metrizable spaces with ordinary metrics d_X and d_Y respectively. It is well-known that $f : X \to Y$ is continuous if and only if for given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$d_X(x, y) < \delta \Rightarrow d_Y(fx, fy) < \epsilon. \tag{2.4}$$

Now let $x, y, z \in X$. Then by (2.5), we have

$$D_X(x, y, z) = \max\{d_X(x, y), d_X(y, z), d_X(z, x)\} < \delta,$$

implies $D_Y(fx, fy, fz) = \max\{d_Y(fx, fy), d_Y(fy, fz), d_Y(fz, fx)\} < \epsilon$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

This shows that the mapping $f : X \to Y$ is continuous w.r.t. D-metrics D_X and D_Y on X and Y, respectively.

Conversely, suppose that the mapping $f : X \to Y$ is continuous w.r.t the D-metrics on X and Y. Then by Theorem 2.1, for $\epsilon > 0$, there exists a $\delta > 0$ such that, $D_X(x, y, z) < \delta$ implies that $D_Y(fx, fy, fz) < \epsilon$. In particular $D_X(x, y, y) < \delta$ implies $D_Y(fx, fy, fy) < \epsilon$. This, further by definitions of D_X and D_Y, implies that $d_X(x, y) < \delta \Rightarrow d_Y(fx, fy) < \epsilon$, showing that $f : X \to Y$ is continuous w.r.t. ordinary metrics on X and Y. This completes the proof.

The continuity of the mappings is closely related to the convergence of the sequence in D-metric spaces. To prove a result in this direction, we need the following definition (Dhage 1984, 1992).

DEFINITION 2.2. *A sequence $\{x_n\}$ of the points of a D-metric space X is said to converge to a point $x \in X$ if for given $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ ($\mathbb{N}$ denotes the set of natural numbers) such that $D(x_m, x_n, x) < \epsilon$ for all $m, n \geq n_0$. In this case we write $x_n \to x$.*

THEOREM 2.2. *Let X and Y be two D-metric spaces with D-metrics D_X and D_Y, respectively. Then the mapping $f : X \to Y$ is continuous if and only if, $x_n \to x \Rightarrow fx_n \to fx$.*

Proof. Suppose f is continuous and let $\{x_n\}$ be a sequence in X converging to a point $x \in X$. Then by Theorem 2.1, for $\epsilon > 0$, there exists a $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$D_Y(fx_m, fx_n, fx) < \epsilon, \text{ whenever } D_X(x_m, x_n, x) < \delta,$$

for all $m, n \geq n_0$. This shows that $fx_n \to fx$ whenever $x_n \to x$.

Conversely suppose that $x_n \to x \Rightarrow fx_n \to fx$. Then for $\epsilon > 0$, there exist a $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0, D_x(x_m, x_n, x) < \delta \Rightarrow D_Y(fx_m, fx_n, fx) < \epsilon$. By Theorem 2.1, f is continuous and the proof is complete.

Finally, in the following it will be shown that the mappings on D-metric spaces satisfying certain Lipschitz type condition are continuous. For, we need the following definition.

DEFINITION 2.3. *Let X and Y be two D-metric spaces with D-metrics D_X and D_Y, respectively. Then a mapping $f : X \to Y$ is called of Lipschitz type if there exists a constant $L > 0$ such that*

$$D_Y(fx, fy, fz) \leq L D_X(x, y, z) \tag{2.5}$$

for all $x, y, z \in X$.

THEOREM 2.3. *If $f : X \to Y$ is a mapping of Lipschitz type, then f is continuous.*

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. We show that the mapping f satisfies the $(\epsilon - \delta)$ condition of Theorem 2.1. Since $f : X \to Y$ is of Lipschitz type, there is a constant $L > 0$ such that

$$D_Y(fx, fy, fz) \leq LD_X(x, y, z)$$

for all $x, y, z \in X$. Let $\epsilon > 0$ be given. Choose $\delta = \frac{\epsilon}{2L} > 0$.

Then by above inequality, we obtain

$$D_Y(fx, fy, fz) \leq LD_X(x, y, z) \leq L\delta < \epsilon,$$

whenever $D_X(x, y, z) < \delta$. By Theorem 2.1, f is continuous on X.

We note that the converese of above Theorem 2.3 is not necessarily true. This will be clear from the following example.

EXAMPLE 2.1. Consider the D-metric space $\mathbb{R}$ with a D-metric D defined by

$$D(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$$

for $x, y, z \in \mathbb{R}$. Define a mapping $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.

Obviously, f is continuous on R with respect to the D-metric D on $\mathbb{R}$, but not a Lipschitz type. For, let $x, y, z \in \mathbb{R}$, then we have

$$\begin{aligned} D(fx, fy, fz) &= D(x^2, y^2, z^2) \\ &\leq L(x, y, z)D(x, y, z) \end{aligned}$$

where $0 < L(x, y, z) = \max\{|x + y|, |y + z|, |z + x|\}$. Since L is not a constant the mapping f is not of Lipschitz type.

As a consequence of Theorem 2.3, we have the following corollary.

COROLLARY 2.2 *Let X and Y be two D-metric spaces with D-metrics D_X and D_Y, respectively, defined by* (2.2) *and* (2.3), *and let $f : X \to Y$ be a mapping of Lipschitz type. Then f is continuous with respect to the oridinary metrics d_X and d_Y on X and Y, respectively.*

Proof. Since f is of Lipschitz type, it is continuous w.r.t. the D-metrics D_X and D_Y on X and Y, respectively. Now, the conclusion follows from Corollary 2.1. The proof is complete.

3. Conclusion. The D-metric spaces are the generalizations of the ordinary metric spaces in some sense, i.e., for a given ordinary metric space, we can find the corresponding associated D-metric spaces with the same non-empty sets. So, for a given continuous mapping

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

on a ordinary metric space, we can find the corresponding D-metric space on which it is continuous. Therefore, from the foregoing discussion, it is clear that our continuity theorems for the mappings in D-metric spaces are more general than in ordinary metric spaces.

The continuity theorems for the mappings in D-metric spaces seem to have applications to the fixed point theory in D-metric spaces and to the approximation theory concerning the existence theorem for the mutually nearest points of three sets which are yet to be investigated.

Acknowledgement. The author is thankful to the referee for giving comments on the earlier version of this paper. He is also thankful to Prof. Dr. S. Gahler (Germany) for providing the reprints of his papers.

References

Dugundji, J. (1975) : *Topology*, Prentice Hall of India Pvt. Ltd., New Delhi.

Dhage, B. C. (1984) : A Study of Some Fixed Point Theorems, *Ph.D. Thesis, Marathawada Univ.* Aurangabad (India).

— (1992) : Generalized metric spaces and mappings with fixed point, *Bull. Cal. Math. Soc. 84(4)*, 329.

— : Generalized metric spaces and topological structure I, *Math. Nachr.* (to appear).

— (1994) : Generalized metric spaces and topological structure II, *Pure Appl. Math. Sci., 40*, 1, 37.

Gahler, S. (1963) : 2-metrische raume und ihre topologische structure, *Math. Nachr. 26*, (1963/64), 115.

Munkres, J. R. (1984) : *Topology–A First Course*, Prentice Hall of India Pvt. Ltd., New Delhi.

GURUKUL COLONY
AHMEDPUR 413515
DISTRICT LATUR
MAHARASHTRA
INDIA

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc., *86*, 509–520 (1994)

A SURVEY OF CLASSES OF SELF-SCALING VARIABLE METRIC ALGORITHMS

M. AL-BAALI

(*Received 3 June 1993; revision received 13 April 1994*)

Abstract. In this paper we consider minimizing a certain measure for self-scaling variable metric algorithms. We show that the minimization yields an explicit formula for a new class of self-scaling algorithms. Experiments with some algorithms are provided. We summarize some numerical results which are required to solve a set of standard test problems. We also show that the self-dual formula of Oren and Spedicato is a member of a new class, which defines a unique minimizer of a measure of the Broyden class, in a sense to be defined.

1. Introduction. In this paper we consider self-scaling variable metric line search algorithms for minimizing a differentiable function $f(x)$, where $x \in R^n$. Each algorithm generates a sequence of points $\{x^{(k)}\}_{k=1}^{\infty}$ iteratively. To initiate the sequence, a starting point $x^{(1)}$ has to be set by the user of the algorithm which begins by testing $\nabla f(x^{(k)})(= g(k)$, say), the gradient vector. If it is zero, the algorithm terminates, but otherwise a new estimate of a local solution, x^*, say, is computed as follows. A search direction $s^{(k)}$ is calculated by solving the set of equations

$$B^{(k)}s^{(k)} = -g^{(k)}. \tag{1.1}$$

where $B^{(k)}$ is a symmetric matrix which is chosen to be some matrix that approximates $\nabla^2 f(x^{(k)})$, the Hessian matrix at $x^{(k)}$. The next estimate of x^* is defined by the line search

$$x^{(k+1)} = x^{(k)} + \alpha^{(k)}s^{(k)}, \tag{1.2}$$

where $\alpha^{(k)}$ (a steplength along $s^{(k)}$) is chosen to satisfy certain standard conditions (Al-Baali and Fletcher, 1986) for which

$$\delta^{(k)T}\gamma^{(k)} > 0, \tag{1.3}$$

where

$$\delta^{(k)} = x^{(k+1)} - x^{(k)} \tag{1.4}$$

---

* This paper is a revised version of the report "Partial self-scaling variable metric algorithms", Department of Systems, Report No. 80(1988), University of Calabria.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

and

$$\gamma^{(k)} = g^{(k+1)} - g^{(k)}. \tag{1.5}$$

In the methods which we are investigating, $B^{(k)}$ is positive definite which ensures that $s^{(k)}$ is a descent direction. The initial Hessian approximation $B^{(1)}$ is defined by the user (usually, $B^{(1)} = I$, the identity matrix) and $B^{(k)}$, for $k > 1$, is determined by an updating operation on a certain matrix. For example, the BFGS method defines $B^{(k+1)}$ by

$$B^{(k+1)} = \text{bfgs}\,(B^{(k)}), \tag{1.6}$$

where, for any symmetric matrix $\hat{B}$,

$$\text{bfgs}\,(\hat{B}) = \hat{B} - \frac{\hat{B}\delta^{(k)}\delta^{(k)T}\hat{B}}{\delta^{(k)T}\hat{B}\delta^{(k)}} + \frac{\gamma^{(k)}\gamma^{(k)T}}{\delta^{(k)T}\gamma^{(k)}}, \tag{1.7}$$

the popular BFGS update of $\hat{B}$. This update has an important property which is that the resulting matrix bfgs $(\hat{B})$ is positive definite if $\hat{B}$ is positive definite (assuming inequality (1.3) holds). Therefore, the BFGS method maintains $B^{(k)}$ matrices positive definite for all k.

The BFGS update (1.6) is a member of the Broyden class of updates, defined by

$$B^{(k+1)} = \text{bfgs}\,(\hat{B}_\theta), \tag{1.8}$$

where

$$\hat{B}_\theta = B + \theta w w^T, \tag{1.9}$$

$$w = (\delta^T B\delta)^{1/2}\left(\frac{\gamma}{\delta^T\gamma} - \frac{B\delta}{\delta^T B\delta}\right), \tag{1.10}$$

θ is a real parameter and the superscript "$^{(k)}$" is suppressed throughout the paper on δ, γ and B. Observe that the choices $\theta = 0$ and $\theta = 1$ give the well known BFGS and DFP updates, respectively (Fletcher, 1987). [Note that the notation H^*_ϕ will be used to refer to the Broyden class of inverse updates. It is defined by the dual of (1.8), that is the right hand side of (1.8), but with bfgs replaced by dfp (the DFP update) and with the interchanges $\delta \leftrightarrow \gamma, B \leftrightarrow H$ and $\theta \leftrightarrow \phi$, in (1.9-10), where ϕ is a parameter and H is used throughout to refer to B^{-1}. For further details see Fletcher, 1987, for instance.]

To ensure that $B^{(k)}$ matrices remain positive definite, the above property for the BFGS update suggests that the matrix $\hat{B}_\theta$ should be chosen positive definite which is possible by choosing values for θ for which

$$\theta > -\frac{1}{w^T H w} \equiv \bar{\theta}. \tag{1.11}$$

(It is clear that this inequality holds for $\theta \geq 0$, since H is positive definite.) We motivate condition (1.11) by continuity and the observation that $\hat{B}_\theta$ is singular at $\theta = -1/w^T H w$, since its inverse is given by

$$\hat{H}_\theta \equiv \hat{B}_\theta^{-1} = H - \frac{\theta H w w^T H}{1 + \theta w^T h W}. \tag{1.12}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

However, letting b and h be denoted by

$$b = \frac{\delta^T B \delta}{\delta^T \gamma}, \quad h = \frac{\gamma^T H \gamma}{\delta^T \gamma} \tag{1.13}$$

btain, using (1.10), that

$$a \equiv w^T H w = bh - 1 \tag{1.14}$$

h shows that (1.11) is equivalent to the usual presentation $\theta > -1/(bh-1)$ (Fletcher,).

It is worth mentioning that if $a = 0$, the Broyden class reduces to a single update (Al-li, 1993a). That paper also provides two ways for motivating the Broyden class of updates. tly, the matrix $B_\theta^{*-1} = B^{(k+1)-1}$, given by the inverse of (1.8), solves the problem

$$\min_{H^*} ||H^* - \hat{H}_\theta||_W \tag{1.15}$$
$$\text{subject to} \quad H^*\gamma = \delta,$$

re W is any symmetric positive definite matrix for which $W\delta = \gamma$. The weighted norm here efined for any matrix E by

$$||E||_W = [\text{trace}(WEWE)]^{1/2}.$$

Secondly, the solution of the dual problem of problem (1.15),

$$\min_{B^*} ||B^* - \bar{B}_\phi||_V \tag{1.16}$$
$$\text{subject to} \quad B^*\delta = \gamma,$$

ere $\bar{B}_\phi$ is the dual of (1.12) and V is any symmetric positive definite matrix for which $V\gamma = \delta$, iven by H_ϕ^{*-1}. In the following we assume that $H_\phi^* = B_\theta^{*-1}$ in which

$$\phi = \frac{1-\theta}{1+\theta a}. \tag{1.17}$$

In order to define our measure below, let Δ be a measure for the Broyden class (or its erse) for any W and V. Al-Baali (1993a) suggests that

$$\Delta(\theta, W, V) = ||H_\phi^* - \hat{H}_\theta||_W^2 + ||B_\theta^* - \bar{B}_\phi||_V^2,$$

sum of the squares of the corresponding two minimal of the above problems. Choosing $= V^{-1} = B_\theta^*$, we obtain, using Al-Baali's (1993a) result, that

$$\begin{aligned}\Omega(\theta, b, h) &\equiv \Delta(\theta, B_\theta^*, H_\phi^*) \\ &= \frac{1}{b^2}\left(\frac{bh+\theta a}{1+\theta a}\right)^2 + \frac{1}{b^2h^4}(\theta a^2 + a + bh)^2 - \frac{2}{b} - \frac{2}{h} + 2\end{aligned} \tag{1.18}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

which we will consider as a measure for the Broyden class. For example, substituting $\theta = 0$ in (1.18) it follows that

$$\Omega(0, b, h) = h^2 + \frac{1}{b^2h^4}(2bh - 1)^2 + 2\left(1 - \frac{1}{b} - \frac{1}{h}\right), \tag{1.19}$$

a measure for the BFGS update. This measure plays a desirable role in Section 4 for defining new "partial" self-scaling algorithms. To define the self-scaling Broyden class of updates, τ is introduced to the Broyden class in order to prescale B-matrix before updating (i.e., τB replaces B into (1.8-10)) in which

$$B^{(k+1)} = \text{bfgs}\,[\tau(B + \theta w w^T)]. \tag{1.20}$$

This update is a subset of the class of Huang (1970) and is known as self-scaling variable metric class of updates (Oren, 1974).

Using the above replacement that $B \to \tau B$ (which implies from (1.13) that $b \to \tau b$ and $h \to h/\tau$), it follows, using (1.10), that condition (1.11) is unchanged. Thus, (1.11) (i.e., $\theta > \bar{\theta}$) also ensures $B^{(k+1)}$ in (1.20) be positive definite. Using the same replacement, it follows from (1.14) and (1.18) that

$$\Omega(\theta, \tau b, h/\tau) = \frac{1}{\tau^2 b^2}\left(\frac{bh + \theta a}{1 + \theta a}\right)^2 + \frac{\tau^2}{b^2h^4}(\theta a^2 + a + bh)^2 - \frac{2}{\tau b} - \frac{2\tau}{h} + 2 \tag{1.21}$$

which we will use as a measure for the self-scaling class (1.20).

In this paper we consider minimizing this measure in three possibilities with respect to either θ, τ or (θ and τ) subject to the so called Oren's conditions

$$0 \le \theta \le 1 \tag{1.22}$$

$$\text{and} \quad \frac{1}{b} \le \tau \le h. \tag{1.23}$$

In Section 2 we minimize (1.21) with respect to either θ or τ and consider the other parameter fixed (i.e., given). These cases yield two classes of self-scaling algorithms (one of them can be used in practice). In Section 3, we give new approaches for deriving the self-dual self-scaling formula of Oren and Spedicato (1976). In one approach we show that this formula defines the minimizer, over θ and τ, of Ω. In Section 4, we illustrate that scaling $B^{(k)}$ at every iteration do not yield a useful method. Thus, we will suggest criterion tests to specify $B^{(k)}$ matrices so that scaling them would be desirable. We test these criterions by applying some methods to a set of standard test problems. The numerical results are summarized in Section 5.

2. Classes of Self-Scaling Variable Metric Algorithms. Oren and Spedicato (1976) suggest a scalar measure function of the two variables θ and τ (which we refer to as

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$\theta, \tau))$ for a bound on the condition number of the updated matrix in the self-scaling class $0)$. For $a \neq 0$, the authors minimize Φ with respect to θ (for given τ) to obtain the class

$$\tau = \frac{h}{1 + a\theta}. \tag{2.1}$$

This relationship is also valid for $a = 0$ due to the following. We repeat the latter , but by fixing θ rather than τ and minimizing Φ with respect to τ. We obtained, after a e manipulation, equation (2.1). It is clear that if $\theta \in [0, 1]$, then it follows from (2.1) that $[1/b, h]$. Thus, Oren's conditions (1.22–23) hold. Oren and Spedicato suggest the choice

$$\tau^* = \left(\frac{h}{b}\right)^{1/2}, \quad \theta^* = \frac{1}{1 + \sqrt{bh}} \tag{2.2}$$

define τ and θ, respectively, in (1.20), which defines a self-dual self-scaling update (which s some desirable properties as will be seen in Section 3).

In this section we repeat the above two ideas, but on the basis of minimizing the measure $\Omega(\theta, \tau b, h/\tau)$, defined by (1.21), rather than Φ. Firstly, we consider τ fixed and nimize over θ the function Ω subject to $\theta > \bar{\theta}$. Assuming $a \neq 0$ and differentiate Ω with pect to θ, it follows after a little manipulation that

$$\frac{\partial \Omega}{\partial \theta} \equiv 2\left(\frac{a}{\tau b}\right)^2 \left[\left(\frac{\tau}{h}\right)^4 (\theta a^2 + a + bh) - \frac{bh + \theta a}{(1 + \theta a)^3}\right] = 0. \tag{2.3}$$

Al-Baali (1986) considers solving the quartic equation (2.3) numerically for $\tau = 1$, which a member of the Broyden class (1.8) is obtained, which was applied to certain test oblems. The author observed that the method works a little worse than the BFGS method some cases. Therefore, the choice $\tau = 1$ is not desirable. It is worth mentioning that the thor also shows that the root of (2.3) which satisfies $\theta > \bar{\theta}$ is unique and, therefore, (2.3) can rearranged as follows

$$\tau = h\left[\frac{bh + \theta a}{(1 + \theta a)^3 (\theta a^2 + a + bh)}\right]^{1/4}. \tag{2.4}$$

Using class (2.4) we define, for instance, a self-scaling BFGS update by substituting $= 0$ in (2.4) in which τ becomes

$$\tau_o = h\left(2 - \frac{1}{bh}\right)^{-1/4} \tag{2.5}$$

bserve that $\tau_o \leq h$). We observe that an optimal value is defined a prior (e.g., given τ by 2.5) then $\theta = 0$ defines the minimizer of $\Omega(\theta, \tau_o b, h/\tau_o)$. As for the class (2.1), if $a = 0$ then it llows from (2.4) that $\tau = h$. Thus, we assume that (2.4) is also valid for $a = 0$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

We now show that class (2.4) satisfies Oren's conditions (1.22–23).

THEOREM 2.1. *If θ is chosen to satisfy* (1.22) *then τ defined by* (2.4) *satisfies* (1.23).

Proof. Because $bh \geq 1$, it follows from (1.14) and (1.22) that

$$bh \leq bh + \theta a \leq 2bh - 1$$
$$1 \leq 1 + \theta a \leq bh$$
$$2bh - 1 \leq \theta a^2 + 2bh - 1 \leq b^2h^2.$$

Hence it follows from (2.4) that $\tau \in [1/b, h]$.

We now consider the second technique that we assume θ fixed and define τ by minimizing over τ the measure $\Omega(\theta, \tau b, h/\tau)$ subject to $\tau > 0$. The minimizer is a root of the quartic equation

$$\frac{1}{2}\frac{\partial\Omega}{\partial\tau} \equiv -\frac{1}{\tau^3 b^2}\left(\frac{bh+\theta a}{1+\theta a}\right)^2 + \frac{1}{\tau^2 b} + \frac{\tau}{b^2h^4}(\theta a^2 + a + bh)^2 - \frac{1}{h} = 0 \tag{2.6}$$

which cannot be solved either for τ or for θ. Thus, class (2.6) is not applicable in practice. However, a note of feature is that $\theta = \theta^*$ and $\tau = \tau^*$ satisfy (2.6).

We end this section by the observation that scaling the Hessian approximation $B^{(k)}$ before updating, for all iterations is not useful as observed in practice, except for a special type of problems.

Although Shanno and Phua (1978) suggest prescaling only $B^{(1)}$ before updating we observed in practice that it is also desirable to prescale several $B^{(k)}$ matrices. Therefore, further study is considered in Sections 4 and 5.

3. Variational Approaches for the Self-Dual Self-Scaling Formula and Features. In this section we give some new features for the self-dual self-scaling formula defined by choice (2.2). We start with the possibility of combining features of both classes (2.1) and (2.4) in the following way. Equating the right hand sides of (2.1) and (2.4) and solving for θ, we obtain after a little manipulation two values for θ. Only one of them is greater than $\bar{\theta}$, defined by θ^* in (2.2). Hence the corresponding value τ^* in (2.2) follows from substituting $\theta = \theta^*$ in (2.1) (or (2.4)). Thus, θ^* is the minimizer over θ of both $\Omega(\theta, \tau^* b, h/\tau*)$ and $\Phi(\theta a, \tau^*)$ which are discussed in Sections 1 and 2, respectively. We also observe that θ^* and τ^* satisfy equation (2.6). Therefore, τ^* is the minimizer over τ of both $\Phi(\theta^* a, \tau)$ and $\Omega(\theta^*, \tau b, h/\tau)$. These results show that the self-dual self-scaling formula shares features of the Oren and Spedicato class (2.1) and our new classes (2.4) and (2.6).

We now show that choice (2.2) defines the minimizer of the two variables function $\Omega(\theta, \tau b, h/\tau)$, defined by (1.21).

THEOREM 3.1. *The least value of Ω is given at point $(\theta, \tau) = (\theta^*, \tau^*)$ which is defined by* (2.2).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. If $a = 0$, then it follows from (1.21) and (2.2) that $\Omega(\theta^*, \tau^* b, h/\tau^*) = 0$ which is the least possible value for Ω, since the latter is nonnegative for all θ and τ. Otherwise if $a \neq 0$ then we proceed as follows. Because (2.4) and (2.6) hold for $\theta = \theta^*$ and $\tau = \tau^*$, $\nabla\Omega = 0$ at point (θ^*, τ^*). To complete the proof we show that the Hessian $\nabla^2\Omega$ at the same point is positive definite in the following way. It can be shown that the minimizer over θ of $\Omega(\theta, \tau b, h/\tau)$ subject to $\theta > \bar{\theta}$ for given $\tau > 0$ is unique (see Al-Baali, 1986, for details). Thus, the inequality $\partial^2\Omega/\partial\theta^2 > 0$ holds for all $\theta > \bar{\theta}$. In particular, we obtain at $(\theta, \tau) = (\theta^*, \tau^*)$ after some manipulation that

$$\mu_1 \equiv \frac{\partial^2\Omega}{\partial\theta^2} = 4\left(\frac{a}{bh}\right)^3 (2a + t), \tag{3.1}$$

$$\mu_2 \equiv \frac{\partial^2\Omega}{\partial\tau^2} = \frac{4}{h^2}\left[\frac{1}{bh}(a+t)^2 - t\right], \tag{3.2}$$

$$\mu_3 \equiv \frac{\partial^2\Omega}{\partial\theta\partial\tau} = \frac{8a^2}{b^2h^3}(a+t), \tag{3.3}$$

$$\text{where} \quad t = \sqrt{bh}. \tag{3.4}$$

It follows from (3.1-4) after some manipulation that

$$\mu_1\mu_2 - \mu_3^2 = cq(t), \tag{3.5}$$
$$\text{where} \quad q(t) = 3t^3 - 4t + 2$$

and $c = 16a^3/t^7 > 0$. From (3.4) we observe that $t \geq 1$ which implies that $q(t) > 0$ (note that $q(t) > 0$ for $t > t_o$, where t_o is some value satisfies the inequality $t_o < -1$). Thus, $\mu_1 > 0$ and $\mu_1\mu_2 - \mu_2^3 > 0$. Hence, the proof is complete.

This result shows particularly that (2.2) defines an optimal self-dual self-scaling update and the corresponding minimal value of Ω follows from (1.21) and (2.2) which can be written, using (3.4), as

$$\Omega^* \equiv \Omega(\theta^*, \tau^* b, h/\tau*) = 2\left[2\left(1 - \frac{1}{t^3}\right) - \frac{1}{t^2}\left(1 - \frac{1}{t^2}\right)\right]$$
$$\leq 4, \tag{3.6}$$

since $t \geq 1$. Inequality (3.6) is the basis for proposing, in the next section, a new self-scaling algorithm.

Finally, it is worth mentioning that it is possible to obtain the self-dual self-scaling formula as follows. Assuming that the dual of the parameter θ is unchanged (i.e., $\phi = \theta$), then it follows from (1.17) that

$$\theta(1 + \theta a) = 1 - \theta.$$

Solving this quadratic equation subject to $\theta > \bar{\theta}$ we obtain the unique root $\theta = \theta^*$. (This result is also observed by R. Fletcher, private communication.) Now, substituting the latter equation in (2.1) (or (2.4)), we obtain $\tau = \tau^*$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

4. Self-Scaling Techniques. Shanno and Phua (1978) show that prescaling $B^{(k)}$ matrix at each iteration before updating is not the reason for obtaining good numerical results when self-scaling methods applied to the extended type of problem. The authors observed that those good results are also obtained if only $B^{(1)}(= I$, the identity matrix) is prescaled before updating at the first iteration by $\tau = h$ and the usual (i.e., unscaled) BFGS update is used for other iterations. Other researchers have considered certain possibilities for defining $B^{(1)}$, but the latter choice (or similarly $B^{(1)} = \tau I$, for some τ) is preferable (R. B. Schnabel, private communication, e.g., Dennis and Schnabel, 1983).

In practice, we generally observed that when the standard initial point was used to define $x^{(1)}$, prescaling only $B^{(1)}$ is preferable to prescaling $B^{(k)}$, for all k. A useful numerical feature of prescaling when applied to the extended type of problem is that the number of line searches (hence, the number of function and gradient evaluations) required to solve the problem are hardly increased as the number of variables increased (Shanno and Phua, 1978). However, when $x^{(1)}$ was sufficiently remote away from the solution x^*, we observed that the latter feature does not hold when only $B^{(1)}$ is prescaled. Thus, prescaling only $B^{(1)}$ may not be sufficient and prescaling other matrices could be desirable. To specify the matrix $B^{(k)}$ that requires prescaling for some k, we suggest below criterion tests in which "partial" self-scaling algorithms are defined.

Because BFGS method is preferred from the Broyden class (it is superlinearly convergent under mild conditions, Powell, 1976), we consider the choice $\theta = 0$ in the rest of this section and suggest partial self-scaling BFGS methods.

When either BFGS or certain members of the self-scaling algorithms of Section 2 were applied to a large number of test problems we observed at the first iteration in all cases that

$$\Omega(0, \tau b, h/\tau) \ll \Omega(0, b, h), \tag{4.1}$$

whether $\tau = h$ or $\tau = \tau_o$ which is given by (2.5). (Note that Ω is defined by (1.21) and the right hand side of (4.1) is given by (1.19).) In addition, when $x^{(1)}$ was not the standard point we also observed case (4.1), at some iterations. In these cases, prescaling some matrices $B^{(k)}$ is desirable. However, to use (4.1) in practice we suggest the criterion test

$$\Omega(0, \tau b, h/\tau) < \eta\Omega(0, b, h), \tag{4.2}$$

where $\eta \in (0, 1)$ (typically, η is a little greater than the tolerance precision). Hence, a partial self-scaling BFGS update is defined as follows

$$B^{(k+1)} = \begin{cases} \text{bfgs}\,(\tau B) & \text{if (4.2) holds} \\ \text{bfgs}\,(B) & \text{otherwise.} \end{cases} \tag{4.3}$$

For the choices that η is equal approximately to 100 times the machine epsilon and $\tau = h$ (or $\tau = \tau_o$), we observed in practice the following: If $x^{(1)}$ is the standard initial point, then condition (4.2) usually holds at the first iteration, but if $x^{(1)}$ is sufficiently remote away

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

the solution then condition (4.2) holds at initial few iterations, but does not hold on ers. Thus, assuming this occurrence, we can avoid testing (4.2) at most iterations and gest another partial self-scaling method as follows

$$B^{(k+1)} = \begin{cases} \text{bfgs}\,(\tau B) & \text{if } k \le l \\ \text{bfgs}\,(B) & \text{otherwise.} \end{cases} \tag{4.4}$$

ially, l is defined sufficiently large (e.g., let l be the maximum number of iterations, provided the user). We begin by testing (4.2) at each iteration. If (4.2) holds, l is unchanged (i.e., t large). Otherwise if (4.2) does not hold for some k, then we set $l = k$ (observe that for $= l+1, l+2, \ldots$, we do not test (4.2) and the unscaled BFGS update is employed). This hod has the advantage that it ultimately chooses the BFGS steps for sufficiently small η, ce, converges superlinearly, for convex objective functions. This method with τ is equal to or τ_o) works well. It solved certain problems that were not solved by BFGS (with or without ling the initial matrix $B^{(1)}$).

It is worth mentioning that the above partial self-scaling methods are defined for any itive τ (not necessarily restrictive to either $\tau = h$ or $\tau = \tau_o$). (For different choices of τ, Dennis and Schnabel, 1983, for instance.) When the memory is limited (Liu and Nocedal, 8), (4.3) method could be used rather than (4.4), since the information about $B^{(k)}$ are ally break (Fletcher, 1988).

Although the above approaches are based on some heuristic arguments, the numerical ults in Section 5 illustrate that prescaling some $B^{(k)}$ before updating is desirable. However, avoid using test (4.2), we consider the following method. On the basis of (3.6), we refer to region

$$D = \{(\theta, \tau) : \Omega(\theta, \tau b, h/\tau) \le 4\} \tag{4.5}$$

"an optimal" region for the parameters θ and τ. If both parameters belong to D, for me values, we use them to define a self-scaling update, but otherwise, we use the self-dual f-scaling update which ensures that θ and τ belong to D.

The latter approach insures that $(\theta, \tau) \in D$ for all iterations. For example, letting $\theta = 0$ d $\tau = 1$, a method which switches between the BFGS update and the self-dual self-scaling date can be defined by

$$B^{(k+1)} = \begin{cases} \text{bfgs}\,[\tau^*(B + \theta^* w w^T)] & \text{if } \Omega(0, b, h) > 4 \\ \text{bfgs}\,(B) & \text{otherwise} \end{cases} \tag{4.6}$$

In a manner similar to that of deriving (4.4) from (4.3), we modify (4.6) as

$$B^{(k+1)} = \begin{cases} \text{bfgs}\,[\tau^*(B + \theta^* w w^T)] & \text{if } k \le l \\ \text{bfgs}\,(B) & \text{otherwise.} \end{cases} \tag{4.7}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Initially, l is defined sufficiently large. At each iteration, we test condition $\Omega(0, b, h) \leq 4$. If it holds for some k, we set $l = k$. Thus, for all $k > l$, the latter condition is not tested and the unscaled BFGS update is used.

In practice, generally, we observed that Method (4.6) is less efficient than the unscaled BFGS method, while Method (4.7) with $l = 1$ works better than the unscaled BFGS method. Thus, Method (4.6) is not useful in practice unless some conditions on τ are made. For example, replacing τ^* by $\min(\tau^*, 1)$, as suggested for self-scaling methods by Al-Baali (1993b), improves the performance of Method (4.6).

5. Computational results. To give some idea about the performance of the partial self-scaling algorithms, defined in Section 4, we give some numerical results in Tables 1 and 2. BFGS is referred to the unscaled BFGS method with $B^{(1)} = I$ (the identity matrix), while BFGS1 is the same as BFGS, except that $B^{(1)}$ is prescaled before updating by h (which is equivalent to (4.4) with $l = 1$ and $\tau = h$). $(4.3)_1$ and $(4.4)_1$ are referred with $\tau = h$ to (4.3) and (4.4) algorithms, respectively. For the latter two algorithms, we prescale the initial matrix same as BFGS1 whether test (4.2), for $k = 1$, holds or not (this technique makes the comparison with BFGS1 reasonable). We chose $\eta = 10^{-7}$ to obtain the results in Table 1 and $\eta = 10^{-8}$ to obtain the results in Table 2. These choices for η were used due to the fact that we run the algorithms on two different personal computers (Commodore 64 and Olivetti with relative precision approximately 1.6×10^{-10} and 5.6×10^{-8}, respectively). Because τ might become very large (usually at the first iteration) which happened very rarely, in which overflow may occur, we set $\tau = \min(h, 10^7)$ rather than $\tau = h$.

The results in Table 1 were obtained by applying the BFGS, BFGS1, $(4.3)_1$ and $(4.4)_1$ algorithms to a set of 51 standard test problems. Most of the tests were taken from the paper of Moré et al (1981) (see Al-Baali, 1986, for details). The columns headed CLS, CNF and CNG denote the commutative numbers of line searches, numbers of function evaluations and numbers of gradient evaluations, respectively, required to solve the test problems in the set. It can be seen that there is little to choose among the partial self-scaling algorithms, but all of them work better than the BFGS algorithm.

Table 1 Results for (5.1) test problems

Algorithm	CLS	CNF	CNG
BFGS	2056	5058	4464
BFGS1	1658	4318	3856
$(4.3)_1$	1633	4210	3769
$(4.4)_1$	1631	4202	3765

To show that BFGS1 algorithm could be inferior to the other partial self-scaling algorithms, we chose $x^{(1)}$ sufficiently remote away from x^* and we applied the above methods to

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

he extended Rosenbrock functions

$$f(x) = \sum_{i=1}^{n/2} [100(x_{2i} - x_{2i-1}^2)^2 + (1 - x_{2i-1})^2], \tag{5.1}$$

here n is even and

$$\bar{x} = (-1.2, 1, \ldots, -1.2, 1)^T$$

s the standard initial point. The reason for selecting this type of problem is that the advantages of BFGS1 over BFGS can occur (Shanno and Phua, 1978). For the choices $n = 10, 20$ and $x^{(1)} = \bar{x}, 100\bar{x}$, numerical results are given in Table 2. Each entry in the table contains the numbers of line searches, function evaluations and gradient evaluations (in that order) required to terminate the iteration. The letter "F" denotes failure of the algorithm. It can be seen that BFGS1 works well when $x^{(1)}$ is the standard point, but (4.3) and (4.4) algorithms work well whether $x^{(1)}$ is the standard point or not and, therefore, they are the winners. Observe that the identical numbers in Table 2 indicate that test (4.2) does not hold for $k = l, l+1, \ldots,$ for some l (these identical results do not usually occur for sufficiently large η). Therefore, (4.4) is preferable for most type of problems.

Finally, it is worth mentioning that the above numerical results, even though are not very encouraging, indicate that scaling several matrices (not necessarily the initials) improves the efficiency of the BFGS method. To know these matrices further research is required to propose a criterion test together with that defined by (4.5). See also Al-Baali (1993b) who obtained desirable theoretical and numerical results by ensuring that $\tau \leq 1$.

Table 2 Applications to the extended Rosenbrock (5.1) problem

$x^{(1)}$	$\bar{x}$		$100\bar{x}$	
n	10	20	10	20
BFGS	52 119 105	86 185 67	249 616 550	F
BFGS1	21 58 50	22 59 51	F	F
$(4.3)_1$	21 58 50	22 59 51	83 231 200	88 235 209
$(4.4)_1$	21 58 50	22 59 51	83 231 200	88 235 209

F = Fail.

Acknowledgement. Most of this work was done while the author was visiting Systems Department, Calabria University (Italy) and the International Centre for Theoretical Physics (Italy). The author is grateful to Professor L. Grandinetti (Chairman of the Systems Department) for providing a stimulating atmosphere, helpful discussions and encouragement. Thanks also due to Professor Abdus Salam (Director of the ICTP) and his members of the staff for arranging my visit and the stimulating atmosphere. The author would also like to thank

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Professor R. B. Schnabel, for providing details concerning his experience with the self-scaling algorithms, and thank the anonymous referee for the carefully reading of the paper.

References

Al-Baali, M. (1986) : Optimal variable metric methods for unconstrained optimization, *Maths. Report* DMUDI, Damascus University.

Al-Baali, M. (1993a) : Variational quasi-Newton methods for unconstrained optimization, *JOTA*, 77, 127.

Al-Baali, M. (1193b) : Analysis of a family of self-scaling quasi-Newton Methods, *Tech. Report*, Department of Mathematics and Computer Science, United Arab Emirates University.

Al-Baali, M. and **Fletcher, R.** (1986) : An efficient line search for nonlinear least squares, *JOTA*, 48, 359.

Dennis, J. E. and **Schnabel, R. B.** (1983) : *Numercial methods for unconstrained optimization and nonlinear equations*, Prentice-Hall, England Cliffs, NJ.

Fletcher, R. (1987) : *Practical methods of optimization*, Chichester, England, Wiley.

Fletcher, R. (1988) : Low storage methods for unconstrained optimization, *NA report* 117, Dundee University.

Huang, H. Y. (1970) : Unified approach to quadratically convergent algorithms for function minimization, *JOTA*, 5, 405.

Liu, D. C. and **Nocedal, J.** (1988) : 'On the limited memory BFGS method for large scale optimization", *Math. Prog.* 45, 503.

More, J. J.; Garbow, B. S. and **Hillstrom, K. E.** (1981) : 'Testing unconstrained optimization software', *ACM transactions on mathematical software*, 7, pp. 17.

Oren, S. S. (1974) : On the selection of parameters in self-scaling variable metric algorithms, *Math. Prog.*, 7, 351.

Oren, S. S. and **Spedicato, E.** (1976) : Optimal conditioning of self-scaling variable metric algorithms, *Math. Prog.*, 10, 70.

Powell, M. J. D. (1976) : Some global convergence properties of a variable metric algorithm for minimization without exact line searches, In Nonlinear programming (R. W. Cottle and C. E. Lemke, Eds.) *SIAM-AMS* proceeding IX Philadelphia.

Shanno, D. F. and **Phua, K. H.** (1978) : 'Matrix conditioning convergent algorithms for function minimization", *Math. Prog.* 14, 149.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
UNIVERSITY OF DAMASCUS
DAMASCUS
SYRIA

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
UAE UNIVERSITY
UNITED ARAB EMIRATES
DEIS–DIPARTIMINTO DI ELETTRONICA
INFORMATICA E SISTEMISTICA
UNIVERSITA DELLA CALABRIA 87036
RENDE-COSENZA
ITALIA

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

ll. Cal. Math. Soc., 86, 521–526 (1994)

ON PSEUDO W_2-SYMMETRIC MANIFOLDS

U. C. DE AND J. C. GHOSH

(*Received 5 August, 1993*)

Abstract. The object of the present paper is to study Pseudo W_2-symmetric manilds.

Introduction. In a recent paper Chaki (1987) introduced and studied a type of on-flat Riemannian manifold $(M^n, g)(n \geq 2)$ whose curvature tensor R satisfies the condition

$$(\nabla_X R)(Y,Z,W) = 2A(X)R(Y,Z,W) + A(Y)R(X,Z,W) + A(Z)R(Y,X,W) + A(W)R(Y,Z,X) + g[R(Y,Z,W),X]\rho, \tag{1}$$

here A is a non-zero 1-form

$$g(X,\rho) = A(X), \tag{2}$$

or every vector field X and ∇ denotes the operator of covariant differentiation with respect o the metric g. Such a manifold was called by him a Pseudo symmetric manifold, the 1-form A was called its associated 1-form and an n-dimensional manifold of this kind was denoted by $PS)_n$.

In 1970, G. P. Pokhariyal and R. S. Mishra (1970) introduced a new curvature tensor nd studied its relativistic significance. Such a curvature tensor has been denoted by W_2 and s defined by

$$W_2(X,Y,Z) = R(X,Y,Z) + \frac{1}{(n-1)}[g(X,Z)QY - g(Y,Z)QX)], \tag{3}$$

where Q is the field of symmetric endomorphism corresponding to the Ricci tensor S i.e.,

$$g(QX,Y) = S(X,Y).$$

From (3) we get

$$'W_2(X,Y,Z,U) = 'R(X,Y,Z,U) + \frac{1}{(n-1)}[g(X,Z)S(Y,U) - g(Y,Z)S(X,U)], \tag{4}$$

where

$$'W_2(X,Y,Z,U) = g(W_2(X,Y,Z),U)$$
$$\text{and} \quad 'R(X,Y,Z,U) = g(R(X,Y,Z),U).$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The present paper deals with a type of non-flat Riemmanian manifold $(M^n, g)(n \geq 2)$ whose W_2 curvature tensor satisfies the condition

$$(\nabla_X W_2)(Y,Z,U) = 2A(X)W_2(Y,Z,U) + A(Y)W_2(X,Z,U) + \\ + A(Z)W_2(Y,X,U) + A(U)W_2(Y,Z,X) + g[W_2(Y,Z,U),X]\rho, \tag{5}$$

where A is a non-zero 1-form and ρ and ∇ have the meanings already mentioned. Such a manifold shall be called a Pseudo W_2-symmetric manifold. A shall be called its associated 1-form and an n-dimensional manifold of this kind shall be denoted by $(PW_2S)_n$. It is easy to see that a $(PW_2S)_n$ is not necessarily a $(PS)_n$. It is, therefore, meaningful to undertake the study of a $(PW_2S)_n$. In Section 1 of this paper it is shown that r/n is an eigen value of the Ricci tensor S and ρ, defined by (2), is an eigen vector corresponding to this eigen value where r is the scalar curvature.

A necessary and sufficient condition for a $(PW_2S)_n$ is of zero scalar curvature is obtained. Next it is shown that if in a $(PW_2S)n$ the Ricci tensor vanishes, then it is a $(PS)_n$. In Section 2, an Einstein $(PW_2S)_n$ has been studied. In this section a necessary and sufficient condition for an Einstein $(PW_2S)_n$ to be a $(PS)_n$ has been obtained. In the concluding part it has been shown that if an Einstein $(PW_2S)_n$ admits a unit parallel vector field then it is a $(PS)_n$.

1. Differentiating (4) covariantly we get

$$(\nabla_U W_2)(X,Y,Z) = (\nabla_U R)(X,Y,Z) + \frac{1}{(n-1)}[g(X,Z)(\nabla_U Q)(Y) \\ - g(Y,Z)(\nabla_U Q)(X). \tag{1.1}$$

On contraction of (1.1) gives

$$(\mathrm{div}W_2)(X,Y,Z) = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \\ + \frac{1}{n-1}[g(X,Z)\mathrm{div}(Q)(Y) - \mathrm{div}(Q)(X)g(Y,Z)], \tag{1.2}$$

where div denotes divergence.

Again, contracting (5) we get

$$(\mathrm{div}W_2)(Y,Z,U) = 2A(W_2(Y,Z,U)) + A(Y)\left[\frac{n}{n-1}S(Z,U) - \frac{r}{n-1}g(Z,U)\right] \\ + A(Z)\left[-\frac{n}{n-1}S(Y,U) + \frac{r}{n-1}g(Y,U)\right] + A(W_2(Y,Z,U));$$

where r is the scalar curvature.

$$\text{i.e.,}\quad (\mathrm{div}W_2)(Y,Z,U) = 3A(W_2(Y,Z,U) + A(Y)\left[\frac{n}{n-1}S(Z,U) - \frac{r}{n-1}g(Z,U)\right] \\ + A(Z)\left[-\frac{n}{n-1}S(Y,U) + \frac{r}{n-1}g(Y,U)\right]. \tag{1.3}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Writting X, Y and Z for Y, Z and U we get

$$(\text{div } W_2)(X,Y,Z) = 3A(W_2(X,Y,Z)) + A(X)\left[\frac{n}{n-1}S(Y,Z) - \frac{r}{n-1}g(Y,Z)\right] + A(Y)\left[-\frac{n}{n-1}S(X,Z) + \frac{r}{n-1}g(X,Z)\right] \quad (1.3)$$

From (1.2) and (1.3) it follows that

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) + \frac{1}{n-1}[g(X,Z)\text{div}(Q)(Y) - \text{div}(Q)(X)g(Y,Z)]$$
$$= 3A(W_2(X,Y,Z)) + A(X)\left[\frac{n}{n-1}S(Y,Z) - \frac{r}{n-1}g(Y,Z)\right]$$
$$+ A(Y)\left[-\frac{n}{n-1}S(X,Z) + \frac{r}{n-1}g(X,Z)\right]$$

Putting $X = Z = e_i$ in the above expression where $\{e_i\}, i = 1, 2, \cdots, n$ is an orthonor- basis of the tangent space at a point and i is summed for $1 \geq i \geq n$, we get

$$S(Y,\rho) = \frac{r}{n}g(Y,\rho). \quad (1.4)$$

From (1.4) we can state the following theorem:

THEOREM 1. *In a $(PW_2S)_n (n \geq 2), r/n$ is a eigenvalue of the Ricci tensor S and ρ n eigen vector corresponding to this eigen value.*

We now obtain a necessary and sufficient condition that the scalar curvature r may zero in a $(PW_2S)_n$. First we suppose that $r = 0$ in a $(PW_2S)_n$. Then from (1.4) we get $X,\rho) = 0$.

Therefore, from (4) it follows that

$$'W_2(X,Y,Z,\rho) = 'R(X,Y,Z,\rho). \quad (1.5)$$

Next we suppose that in a $(PW_2S)_n$ the relation (1.5) holds.

Then from (4) we get

$$g(X,Z)S(Y,\rho) = g(Y,Z)S(X,\rho), \quad (1.6)$$

ntraction of (1.6) gives

$$(n-1)S(Y,\rho) = 0, \quad \text{or,} \quad S(Y,\rho) = 0.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Hence from (1.4) we get $r = 0$, because $g(X, \rho)$ i.e., $A(X) \neq 0$. This leads to the following theorem:

THEOREM 2. *$A(PW_2S)_n (n \geq 2)$ is of zero scalar curvature if and only if the relation (1.5) holds.*

Now we obtain a sufficient condition for a $(PW_2S)_n$ to be a $(PS)_n$. Suppose the Ricci tensor $S(X, Y) = 0$ in a $(PW_2S)_n$.

Then we get from (4)

$$'W_2(X, Y, Z, U) = 'R(X, Y, Z, U)$$

$$\text{i.e.,} \quad W_2(X, Y, Z) = R(X, Y, Z). \tag{1.7}$$

But in virtue of (5) and (1.7) the relation (1) holds, i.e., the $(PW_2S)_n$ reduces to a $(PS)_n$.

Thus, we have the following theorem:

THEOREM 3. *If in a $(PW_2S)_n$ the Ricci tensor vanishes, then it is a $(PS)_n$.*

2. In this section we consider an Einstein $(\mathbf{PW_2S})_n$.

As Einstein manifold is defined by

$$S(X, Y) = \frac{r}{n} g(X, Y), \tag{2.1}$$

for which the scalar curvature r is constant. From (2.1) we get

$$(\nabla_Z S)(X, Y) = 0, \tag{2.2}$$

since r is constant.

From Binanchi identity and (2.2) we get

$$\text{div} R = 0. \tag{2.3}$$

By virtue of (2.1) and (2.2) the relation (5) reduces to

$$\begin{aligned}(\nabla_U R)(X, Y, Z) = {} & 2A(U)\left[R(X, Y, Z) - \frac{r}{n(n-1)}\{g(Y, Z)X - g(X, Z)Y\}\right] + \\ & A(X)\left[R(U, Y, Z) - \frac{r}{n(n-1)}\{g(Y, Z)U - g(U, Z)Y\}\right] \\ & + A(Y)\left[R(X, U, Z) - \frac{r}{n(n-1)}\{g(U, Z)X - g(X, Z)U\}\right] \\ & + A(Z)\left[R(X, Y, U) - \frac{r}{n(n-1)}\{g(Y, U)X - g(X, U)Y\}\right] + g(R(X, Y, Z) \\ & - \frac{r}{n(n-1)}[g(Y, Z)X - g(Z, X)Y], U)\rho. \end{aligned} \tag{2.4}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

We now obtain a necessary and sufficient condition for an Einstein $(PW_2S)_n$ to be a $(PS)_n$. First suppose that an Einstein $(PW_2S)_n$ is a $(PS)_n$. It is known (Chaki 1987) that in

$$\mathrm{dr}(X) = 2rA(X) + 4A(QX). \tag{2.5}$$

Since in an Einstein manifold $\mathrm{dr}(X) = 0$ we get from (2.5)

$$r \cdot A(X) = -2A(QX). \tag{2.6}$$

Putting $Y = \rho$ in (2.1) we get

$$S(X, \rho) = \frac{r}{n} g(X, \rho) \tag{2.7}$$

$$\text{i.e.,} \quad A(QX) = \frac{r}{n} A(X)$$

From (2.6) and (2.7) it follows that $r = 0$, since $A(X) \neq 0$.

Next we suppose that in an Einstein $(PW_2S)_n, r = 0$. Then from (2.4) it follows that

$$(\nabla_U R)(X, Y, Z) = 2A(U)R(X, Y, Z) + A(X)R(U, Y, Z) + A(Y)R(X, U, Z) + A(Z)R(X, Y, U) + g(R(X, Y, Z), U)\rho$$

Thus, we have the following theorem:

THEOREM 5. *A necessary and sufficient condition for an Einstein $(PW_2S)_n$ to be a $(PS)_n$ is that the scalar curvature is equal to zero.*

In the concluding part of this section we obtain a sufficient condition for an Einstein $(PW_2S)_n$ to be a $(PS)_n$. For this we require the notion of a parallel vector field which is defined as follows:

A vector field V is said to be parallel (Schouten 1954), (Kobayashi and Nomizu 1963)

$$\nabla_X V = 0. \tag{2.8}$$

We now suppose than an Einstein $(PW_2S)_n$ admits a unit parallel vector field V. Applying Ricci identity to (2.8) we get

$$R(X, Y, V) = 0.$$

From which it follows that

$$S(X, V) = 0. \tag{2.9}$$

Putting $Y = V$ in (2.1) we get

$$S(X, V) = \frac{r}{n} g(X, V). \tag{2.10}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

By virtue of (2.9) we get from (2.10)

$$\frac{r}{n}g(X,V) = 0. \tag{2.11}$$

Again, putting $X = V$ in (2.11) we get $r = 0$, since V is a unit vector. Hence (2.4) reduces to

$$(\nabla_U R)(X,Y,Z) = 2A(U)R(X,Y,Z) + A(X)R(U,Y,Z) + A(Y)R(X,U,Z) + A(Z)R(X,Y,U) + g(R(X,Y,Z),U)\rho$$

This lead to the following theorem:

THEOREM 6. *If an Einstein $(PW_2S)_n$ admits a unit parallel vector field, then it is a $(PS)_n$.*

References

Chaki, M. C. (1987) : On Pseudo symmetric manifolds, Analele Stiintifice Ale Universitatii *'ALI CUZA' Din Iasi*, Romania 33, 53.

Pokhariyal, G. P. and **Mishra, R. S.** (1970): Curvature tensors and their relativistic significance. *Yokohama Math, Journ.* 18, 105.

Schouten, J. A. (1954) : *Ricci calculus* (2nd edition), Springer Verlag, 322.

Kobayashi S. and **Nomizu, K.** (1963) : *Foundations of Differential Geometry*, Vol. 1, Interscience Publishers, 124.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KALYANI
KALYANI 741235
WEST BENGAL
INDIA

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

ll. Cal. Math. Soc., 86, 527–530 (1994)

INCLUSION THEOREMS FOR ABSOLUTE NÖRLUND SUMMABILITY

IKUKO MIYAMOTO

(*Received 18 August, 1993*)

Introduction. Let $\{p_n\}$ be a positive sequence and let $P_n = \sum_{k=0}^{n} p_k$. A series $\sum_{n=0}^{\infty} a_n$ ith its partial sum s_n is said to be summable (N, p_n) to sum s, if $t_n = \sum_{k=0}^{n} p_{n-k} s_k / P_n \to s$ as $\to \infty$, and is said to be absolutely summable (N, p_n), or summable $|N, p_n|$, if $\sum |t_n - t_{n-1}| <$ o. For a positive sequence $\{q_n\}$, we set $Q_n = \sum_{k=0}^{n} q_k$. Throughout this paper, a sequence $\{k_n\}$ s defined formally by $\sum k_n x^n = \sum q_n x^n \Big/ \sum p_n x^n$.

Given two summability methods A and B, we write $A \subset B$ if each series summable A s summable B.

On inclusion relations for ordinary Nörlund summability, the following Hardy's Theo- em (1949, Theorem 23) is well known.

THEOREM A. *If (N, p_n) and (N, q_n) are two regular Nörlund methods such that*

$$\frac{p_n}{p_{n-1}} \le \frac{p_{n+1}}{p_n} (n > 0) \tag{1.1}$$

$$\text{and} \quad \frac{p_n}{p_{n-1}} \le \frac{q_n}{q_{n-1}} (n > n_o) \tag{1.2}$$

then $(N, p_n) \subset (N, q_n)$.

On the other hand, Borwein and Cass (1968, Theorem 3) proved the following:

THEOREM B. *Let (N, q_n) be a regular Nörlund method. If $\{p_n\}$ and $\{q_n\}$ satisfy* (1.1),

$$\frac{p_n}{p_{n-1}} \ge \frac{q_n}{q_{n-1}} (n > 0) \tag{1.3}$$

$$\text{and} \quad p_n = O(q_n) \tag{1.4}$$

then $(N, p_n) \subset (N, q_n)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Concerning inclusion relation for absolute Nörlund summability, Kishore (1970, Theorem 1) proved the following

THEOREM C. *If* $\{p_n\}$ *and* $\{q_n\}$ *satisfy* (1.1)

$$\frac{p_{n+1}}{p_n} \leq \frac{q_{n+1}}{q_n} (n \geq 0) \tag{1.5}$$

$$\Delta q_n = q_n - q_{n+1} \geq 0 \tag{1.6}$$

$$\frac{p_{m+1}^2}{p_m p_{m+2}} \leq \frac{q_{n+1}^2}{q_n q_{n+2}} (0 \leq m \leq n) \tag{1.7}$$

$$\text{and} \quad \Delta p_{m+1} \Delta q_n \leq \Delta p_m \Delta q_{n+1} (0 \leq m \leq n) \tag{1.8}$$

then $|N, p_n| \subset |N, q_n|$.

This is an absolute Nörlund summability analogue of Theorem A. The purpose of this paper is to establish some results for absolute Nörlund summability corresponding to Theorem A and B.

THEOREM 1. *If* $\{p_n\}$ *and* $\{q_n\}$ *satisfy* (1.1)

$$\Delta q_n > 0 \tag{1.9}$$

$$p_o \Delta q_n \leq q_o \Delta p_n (n \geq 0) \tag{1.10}$$

$$\text{and} \quad \Delta p_{n+1} \Delta q_n \leq \Delta q_{n+1} \Delta p_n (n \geq 0) \tag{1.11}$$

then $|N, p_n| \subset |N, q_n|$.

REMARK 1. From (1.10), we may get (1.5), by induction.

THEOREM 2. *If* $\{p_n\}$ *and* $\{q_n\}$ *satisfy* (1.1),

$$\Delta p_n > 0 \tag{1.12}$$

$$p_o \Delta q_n \geq q_o \Delta p_n (n \geq 0) \tag{1.13}$$

$$\Delta p_{n+1} \Delta q_n \geq \Delta q_{n+1} \Delta p_n (n \geq 0) \tag{1.14}$$

$$\text{and} \quad P_n = O(Q_n) \tag{1.15}$$

then $|N, p_n| \subset |N, q_n|$.

REMARK 2. From (1.13), we may get (1.3), by induction.

The author takes this opportunity of expressing her heartfelt thanks to Professor H. Hirokawa for his kind encouragement and valuable suggestions in the preparation of this paper.

2. Preliminary Lemmas.

LEMMA 1. *If* $\sum_{v=0}^{n} |k_{n-v}| P_v = O(Q_n)$ *and there exists a positive integer* N *such that* $k_{m-1}/Q_{n-1} \geq k_m/Q_n$ *whenever* $n \geq m \geq N$, *then* $|N, p_n| \subset |N, q_n|$.

This is due to McFadden (1942, Theorem (2.13)).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

LEMMA 2. *If* $\sum_{v=0}^{n} |k_{n-v}|P_v = O(Q_n)$ *and there exists a positive integer* N *such that* $k_{n-1}/Q_{n-1} \leq k_m/Q_n$, *whenever* $n \geq m \geq N$, *then* $|N, p_n| \subset |N, q_n|$.

LEMMA 3. *If* $P_n = O(Q_n)$ *and there exists a positive integer* N *such that* $\{k_n\}$ *is non-positive and non-decreasing whenever* $m > N$, *then* $|N, p_n| \subset |N, q_n|$.

These are easily proved by the method analogous to that of the proof of McFadden 1942, Theorem (2.19) and Theorem (2.13)).

LEMMA 4. *Under the conditions of Theorem* $1, k_n \geq 0$ *and* $\Delta k_n \geq 0$ *for all* $n \geq 0$.

Proof. We assume, without loss of generality, that $p_o = q_o = 1$. Let us write

$$\frac{1}{p(x)} = \frac{1}{\sum_{n=0}^{\infty} p_n x^n} = 1 - \sum_{n=1}^{\infty} c_n x^n.$$

Using (1.1), by Kaluza's Theorem (1949, Theorem 22), we have $c_n \geq 0$. From the identity $\sum_{n=0}^{\infty} k_n x^n = \sum_{n=0}^{\infty} q_n x^n \left(1 - \sum_{n=1}^{\infty} c_n x^n\right)$, we get

$$k_n = q_n - (c_1 q_{n-1} + c_2 q_{n-2} + \ldots + c_n q_o)(n > 0).$$

By the way, using (1.1) and Remark 1, from the proof of Hardy's Theorem A (1949, Theorem 23), we have $k_n \geq 0 (n \geq 0)$.

Next we show that $\Delta k_n \geq 0 (n \geq 0)$. For the case $n = 0$, we have

$$\begin{aligned} \Delta k_o &= k_o - k_1 = q_o - (q_1 - c_1 q_o) \\ &= (q_o - q_1) + c_1 q_o \geq 0. \end{aligned}$$

For the case $n > 0$, by (1.9), (1.10) and (1.11), we have

$$\begin{aligned} \Delta k_n = k_n - k_{n+1} &= \Delta q_n - (c_1 \Delta q_{n-1} + \ldots + c_n \Delta q_o) + c_{n+1} q_o \\ &= \Delta q_n \left\{ 1 - \left(c_1 \frac{\Delta q_{n-1}}{\Delta q_n} + \ldots + c_n \frac{\Delta q_o}{\Delta q_n} \right) \right\} + c_{n+1} q_o \\ &\geq \Delta q_n \left\{ 1 - \left(c_1 \frac{\Delta p_{n-1}}{\Delta p_n} + \ldots + c_n \frac{\Delta p_o}{\Delta p_n} \right) \right\} + c_{n+1} q_o \\ &= \frac{\Delta q_n}{\Delta p_n} \{ \Delta p_n - (c_1 \Delta p_{n-1} + \ldots + c_n \Delta p_o) \} + c_{n+1} q_o \\ &= -\frac{\Delta q_n}{\Delta p_n} c_{n+1} p_o + c_{n+1} q_o \\ &= c_{n+1} p_o \left(\frac{q_o}{p_o} - \frac{\Delta q_n}{\Delta p_n} \right) \geq 0. \end{aligned}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Hence, we have $\Delta k_n \geq 0 (n \geq 0)$. This proves the Lemma 4.

By the method similar to that of the proof of Lemma 4, we have the following

LEMMA 5. *Under the conditions of Theorem* 2 *except* (1.15), $k_n \leq 0$ *and* $\Delta k_n \leq 0$ *for all* $n > 0$.

3. Proof of Theorems

We shall first prove Theorem 1. Since $\{k_n\}$ is non-negative and non-increasing, by Lemma 4, we have

$$\sum_{v=0}^{n} \frac{|k_{n-v}|P_v}{Q_n} = \sum_{v=0}^{n} \frac{k_{n-v}p_v}{Q_n} = 1$$

and $$\frac{k_{m-1}}{Q_{n-1}} - \frac{k_m}{Q_n} = \frac{k_{m-1} - k_m}{Q_{n-1}} + k_m \left(\frac{1}{Q_{n-1}} - \frac{1}{Q_n} \right) \geq 0 \leq m \leq n \quad (0 \geq m \geq n).$$

Hence, by Lemma 1, we have $|N, p_n| \subset |N, q_n|$, which proves Theorem 1.

We shall next prove Theorem 2. By (1.15) and Lemma 5, the assumptions of Lemma 3 are all satisfied. Hence, by Lemma 3, we have $|N, p_n| \subset |N, q_n|$, which is the required result.

References

Borwein, D. and Cass, F. P. (1968) : Strong Nörlund summability, *Math. Z.* 103, 94.

Hardy, G. H. (1949) : *Divergent sereis,* Clarendon Press, Oxford.

Kishore, N. (1970) : Inclusion and equivalence of absolute Nörlund summability methods, *Indian Jour. Math.* 12. 1.

McFadden, L. (1942) : Absolute Nörlund summability, *Duke Math. Jour.* 9, 168.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCES
CHIBA UNIVERSITY
INAGE-KU
CHIBA 263
JAPAN

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

J. Cal. Math. Soc., 86, 531–536 (1994)

ON PROJECTIVE MOTION IN A KAEHLERIAN SPACE WITH RECURRENT CURVATURE

A. K. SINGH AND U. S. NEGI

(*Received 19 August 1993; revision received 14 December 1993*)

Introduction. An $n(=2m)$ dimensional Kaehlerian space K_n^c is a Riemannian space it admits a structure tensor field F_i^h satisfying the relations (Yano 1965).

$$F_i^h F_j^i = -\delta_j^h, \tag{1.1}$$

$$F_{ij} = -F_{ji}, (F_{ij} = F_i^a g_{aj}) \tag{1.2}$$

$$\text{and} \quad F_{i,j}^h = 0. \tag{1.3}$$

here the comma (,) followed by an index denotes covariant differentiation with respect to the etric tensor g_{ij} of the Riemannian space, i.e., the symmetric connection $\left\{ {h \atop ij} \right\}$.

The Riemannian curvature tensor, R_{ijk}^h, is given by (Yano 1965)

$$R_{ijk}^h = \partial_i \left\{ {h \atop jk} \right\} - \partial_j \left\{ {h \atop ik} \right\} + \left\{ {h \atop ia} \right\} \left\{ {a \atop jk} \right\} - \left\{ {h \atop ja} \right\} \left\{ {a \atop ik} \right\}. \tag{1.4}$$

Where $\partial_i = \partial/\partial x^i$ and $\{x^i\}$ denotes real local coordinates and the Ricci-tensor and the scalar urvature are, respectively, given by $R_{ij} = R_{aij}^a$ and $R = R_{ij} g^{ij}$.

A Kaehler space K_n^c is said to be Kaehlerian recurrent, if the following condition is atisfied (Lal and Singh 1971)

$$R_{ijk,a}^h = \lambda_a R_{ijk}^h, \tag{1.5}$$

Where λ_a is a non-zero Recurrence vector, we shall call such a space $^*K_n^c$ space.

Let us consider an $n(=2m)$ dimensional affinely connected space K_n^c with symmetric onnection $\left\{ {h \atop ij} \right\}$, where $(h, i, j, k, \ldots = 1, 2, 3, 4, \ldots n)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The geodesics $x^h = x^h(t)$ of the space are given by

$$\frac{d^2x^h}{dt^2} + \left\{ {h \atop ij} \right\} \frac{dx^i}{dt} \cdot \frac{dx^j}{dt} = \rho(t) \frac{dx^h}{dt},$$

when an infinitesimal transformation

$$\bar{x}^h = x^h + v^h(x)dt. \tag{1.6}$$

Where v^h denotes components of a covariant vector and dt means an infinitesimal constant, transforms the system of geodesics into the same system, (1.6) is called an infinitesimal projective motion, the necessary and sufficient condition that (1.6) be projective motion in K_n^c is that the Lie-derivative of $\left\{ {h \atop ij} \right\}$, w.r.t (1.6) itself has the form:

$$L_v \left\{ {h \atop ij} \right\} = A_i^h F_j + A_j^h F_i. \tag{1.7}$$

Where $A_i^h = \partial_i x^h$, for a certain non-zero covariant vector F_i.

Projective curvature tensor P_{ijk}^h of the space K_n^c is of the form:

$$P_{ijk}^h = R_{ijk}^h + \Omega_{ijk}^h. \tag{1.8}$$

Where we have put

$$\Omega_{ijk}^h = \frac{1}{n+2}\left[R_{ik}\delta_j^h - R_{jk}\delta_i^h + S_{ik}F_j^h - S_{jk}F_i^h + 2S_{ij}F_K^h\right]. \tag{1.9}$$

Where we have defined $S_{ij} = F_i^a R_{aj}$.

From (1.8), (1.9) and the relation $R_{ij} = R_{ijl}^l$.

We can see with ease that

$$P_{ijl}^l = 0 \quad \text{and} \quad P_{ljk}^l = 0, \tag{1.10}$$

substitution of (1.7) into

$$L_v R_{ijk}^h = \left[L_v \left\{ {h \atop ij} \right\} \right]_{,k} - \left[L_v \left\{ {h \atop ik} \right\} \right]_{,j}$$

gives $L_v R_{ijk}^h = A_j^h F_{i,k} - A_k^h F_{i,j} + A_i^h F_{j,k} - A_i^h F_{k,j}$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

From which by contraction, we have

$$L_v R_{ij} = 0 \tag{1.11}$$

In view of (1.7), the Lie-derivative of $\left\{ {h \atop ij} \right\}$ can be written as

$$L_v \left\{ {h \atop ij} \right\} = v^h_{,ij} + R^h_{ijk} v^k.$$

Equation (1.7) contains $n^2 + 2n$ unknown v^h, v^h_i and F_i. The integrability condition of (1.7) is composed of

$$L_v P^h_{ijk} = 0. \tag{1.12}$$

In an affinely connected Kaehlerian Recurrent space $^*K^c_n$.

Let us try to discuss the existence of Projective motion (1.6) satisfying (1.7). For this purpose, at first we have to assume the condition (1.2). In what follows, we shall find an important property, on P^h_{ijk} holding in admitting Projective motion (1.6) from (1.5), we have $R_{ij,a} = \lambda_a R_{ij}$, then applying this fact to (1.9), we have $\Omega^h_{ijk,a} = \lambda_a \Omega^h_{ijk}$ from which in view of (1.5) we can have at least the essential property on P^h_{ijk} of the form:

$$P^h_{ijk,a} = \lambda_a P^h_{ijk}. \tag{1.13}$$

In the present paper, we have defined and studied projective motion in a Kaehlerian space with recurrent curvature and several theorems have been derived.

2. Projective motion in a Kaehlerian space with recurrent curvature:

Now, from (1.11) and (1.13), we have

$$L_v(P^h_{ijk,a}) = (L_v \lambda_a) P^h_{ijk}. \tag{2.1}$$

Introducing (1.7) into the right hand side of the following result:

$$L_v(P^h_{ijk,a}) - (L_v P^h_{ijk})_{,a} = \left(L_v \left\{ {l \atop ai} \right\} \right) P^h_{ljk}$$
$$- \left(L_v \left\{ {l \atop aj} \right\} \right) P^h_{ilk} - \left(L_v \left\{ {l \atop ak} \right\} \right) P^h_{ijl} + \left(L_v \left\{ {h \atop al} \right\} \right) P^l_{ijk}.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

we have

$$L_v(P^h_{ijk,a}) - (L_v P^h_{ijk})_{,a} = -2F_a P^h_{ijk} - F_i P^h_{ajk} - F_j P^h_{iak} - F_k P^h_{ija} + A^h_a F_l P^l_{ijk}.$$

In view of (1.11), we obtain

$$L_v(P^h_{ijk,a}) = -2F_a P^h_{ijk} + A^h_a F_l P^l_{ijk} - F_i P^h_{ajk} - F_j P^h_{iak} - F_k P^h_{ija}.$$

In this way (2.1) is rewritten as

$$(2F_a + L_v\lambda_a)P^h_{ijk} = A^h_a F_l P^l_{ijk} - F_i P^h_{ajk} - F_j P^h_{iak} - F_k P^h_{ija}. \tag{2.2}$$

In (2.2) making the contraction on h and a and using the relation $P^h_{ijk} = -P^h_{ikj}$,

We have

$$(2F_a + L_v\lambda_a)P^a_{ijk} = nF_a P^a_{ijk} - F_i P^a_{ajk} + F_j P^a_{ika} - F_k P^a_{ija},$$

on account of (1.10), from this result, we get the condition

$$(L_v\lambda_a)P^a_{ijk} = (n-2)F_a P^a_{ijk}. \tag{2.3}$$

Multiplying both sides of (2.2) by F_h and summing over h, we can obtain

$$(2F_a + L_v\lambda_a)F_h P^h_{ijk} = F_a F_l P^l_{ijk} - F_i F_h P^h_{ajk} - F_j F_h P^h_{iak} - F_k F_h P^h_{ija} \tag{2.4}$$

$$\text{or,}\quad (L_v\lambda_a)F_h P^h_{ijk} = -F_a F_h P^h_{ijk} - F_i F_h P^h_{ajk} - F_j F_h P^h_{iak} - F_k F_h P^h_{ija}. \tag{2.5}$$

for $n \geq 3$, (2.3) shows

$$F_h P^h_{ijk} = \frac{1}{(n-2)}(L_v\lambda_l)P^l_{ijk} \tag{2.6}$$

So, with the half of equation (2.6) and neglecting the number factor $(n-2)$, we can conclude from equation (2.5) that,

$$(L_v\lambda_a)(L_v\lambda_l)P^l_{ijk} = -F_a(L_v\lambda_l)P^l_{ijk} - F_i(L_v\lambda_l)P^l_{ajk} - F_j(L_v\lambda_l)P^l_{iak} - F_k(L_v\lambda_l)P^l_{ija}.$$

$$\text{that is,}\quad (L_v\lambda_l)[(L_v\lambda_a)P^l_{ijk} + F_a P^l_{ijk} + F_i P^l_{ajk} + F_j P^l_{iak} + F_k P^l_{ija}] = 0. \tag{2.7}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

But from (2.2), we have

$$F_i P^l_{ajk} + F_j P^l_{iak} + F_k P^l_{ija} = A^l_a F_b P^b_{ijk} - (2F_a + L_v\lambda_a) P^l_{ijk}, \tag{2.8}$$

So, (2.8) may be simplified as

$$F_a P^l_{ijk} L_v \lambda_l = (L_v\lambda_a) F_l P^l_{ijk}, \tag{2.9}$$

sing (2.3) and (2.6) in (2.9), we have

$$F_a = \frac{1}{(n-2)} L_v\lambda_a, (n \geq 3). \tag{2.10}$$

In this way, we get the following:

THEOREM (2.1). *If a $^*K^c_n$-spaces $(n \geq 3)$ admits an infinitesimal projective motion, the motion should be of the form:*

$$\bar{x}^h = x^h + v^h(x)dt, L_v \left\{ \begin{matrix} h \\ ij \end{matrix} \right\} = A^h_i F_j + A^h_j F_i,$$

$$F_j = \frac{1}{n-2}(L_v\lambda_j).$$

THEOREM (2.2). *If $L_v\lambda_i$ denotes a Parallel vector, then a $^*K^c_n$-space admits a general projective motion Iff $L_v R^h_{ijk} = 0$.*

Proof. Let us examine the case, where $L_v\lambda_i$ denotes a parallel vector : $(L_v\lambda_i)_{,j} = 0$ operating L_v(Lie-derivative) on both sides of (1.9), we find $L_v\Omega^h_{ijk} = 0$.

Comparing this with (1.8), we get $L_v P^h_{ijk} = L_v R^h_{ijk}$.

In case of present theory, we have $L_v P^h_{ijk} = 0$.

So, it follows that $L_v R^h_{ijk} = 0$.

But this gives us the parallel (and gradient) property of $L_v\lambda_i$.

THEOREM (2.3). *A necessary and sufficient condition for a projective motion admitted in $^*K^c_n$ space to be an affine motion in the same is that $L_v\lambda_i = 0$.*

Proof. We know that, in order that projective motion become affine motion, it is necessary and sufficient that we have $F_i = 0$, or, $L_v\lambda_i = 0$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

References

Lal, K. B. and **Singh, S. S.** (1971) : On Kaehlerian spaces with recurrent Bochner curvature, tensor, *Acc. Naz. Dei Lin Cei, Rend, 51*, (3–4), 213.

Tachibana, S. (1961) : On infinitesimal conformal and projective transformation of complex *K*-spaces, *Tohoku Math. Journ., 13*, 386.

Yano, K. and **Bochner, S.** (1953) : Curvature and Betti numbers. *Ann. Math. Stud., 32*, 83.

Yano, K. (1965) : *Differential Geometry on complex and almost complex spaces*, Pergamon Press, London.

DEPARTMENT OF MATHEMATICS
H. N. B. GARHWAL UNIVERSITY
CAMPUS TEHRI
TEHRI, GARHWAL 249001 (U.P.)
INDIA

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

ll. Cal. Math. Soc., 86, 537–546 (1994)

EFFECTIVE ELASTIC COEFFICIENTS IN AN INITIALLY STRESSED MEDIUM AND PROPAGATION OF RAYLEIGH WAVES IN SUCH A HALF SPACE BOUNDED BY A RIGID LAYER

S. DEY AND* P. MAHATO**

(*Received 26 August 1993*)

Abstract. The paper gives a mathematical analysis to find out the effective elastic nstants of an initially stressed medium and studies the propagation of Rayleigh waves in an itially stressed half space bounded by a rigid layer at the top. The study reveals that the astic constants are very much affected by the presence of initial stresses and concludes that e presence of rigid boundary enhances the velocity of Rayleigh waves in the medium and metimes the two Rayleigh wave fronts are also available for some initial stresses.

Introduction. The presence of initial stresses in a medium brings about some chan- s in the medium. It changes an isotropic medium into anisotropic and modifies the elastic nstants. In addition to this the initial stresses are observed to affect explicitly the propagation body waves.

This paper gives a complete analysis of the effect of initial stresses on the elastic nstants of the medium and studies the effect of the presence of a rigid boundary on the opagation of Rayleigh waves in an initially stressed half space. The earth is an initially ressed medium and the presence of very hard rock (may be taken as rigid) on the surface of e earth is also observed. The problem has been constructed to discuss the effect of the rigid yer and the initial stresses in the propagation of Rayleigh waves.

The numerical calculations have been done and the results are presented in the form graphs. The graphs show that the elastic constants are much modified by the initial stresses nd the propagation of Rayleigh waves is also much affected by the presence of rigid layer and itial stresses. The presence of rigid layer increases the wave velocity of Rayleigh waves and e initial stresses sometimes cause two Rayleigh wave fronts.

1. Effective Elastic Coefficients in an Initially Stressed Medium. Let an astic isotropic medium having elastic constants λ and μ be subjected to any one or more of e processes such as small creep, unequal thermal distribution over loading, gravity field or dial compressive or tensile forces etc. These forces will develop stresses inside the medium and

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

the body will go into a state of stresses. These stresses developed in the medium are called the initial stresses or pre-stresses.

Let S_{11} and S_{22} be the initial stresses and ϵ_{11}, ϵ_{22} be the initial strain components developed in the medium (two dimensional) along the principal directions. Since the medium was isotropic the initial stress components S_{ij} and the initial strain components ϵ_{ij} are connected by the equations

$$\epsilon_{11} = \frac{1}{E}(S_{11} - \sigma S_{22}); \quad \epsilon_{11} = \frac{1}{E}(S_{22} - \sigma S_{11}), \tag{1}$$

where E and σ are Young's modulus and Poisson's ratio of the medium, respectively, and are given by

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}; \quad \sigma = \frac{\lambda}{2(\lambda + \mu)}.$$

These relations show that if the pre-stress components S_{ij} are known, the pre-strain components ϵ_{ij} may be obtained.

Now suppose that the medium with these initial stresses be deformed and u_1 and u_2 be the displacement components in the medium along the rotated axes. These infinitesimal deformation will generate incremental strains and incremental stresses. The initial stresses present in the body makes the medium orthotropic and hence the relationship between the incremental stress and strain components may be written as

$$S_{11} = B_{11}e_{x_1x_1} + B_{12}e_{x_2x_2}; \quad S_{22} = B_{21}e_{x_1x_1} + B_{22}e_{x_2x_2}; \quad S_{12} = 2Qe_{x_1x_2}, \tag{2}$$

where $e_{x_ix_j} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right), i, j = 1, 2.$

The incremental elastic coefficients B_{ij} and Q of the initial stressed medium has been expressed by Biot [1, p.111] in terms of the original elastic coefficients λ, μ and the initial strain components ϵ_{11} and ϵ_{22} are

$$\begin{aligned} B_{11} &= (2\mu + \lambda)(1 + \epsilon_{11} - \epsilon_{22}); \quad B_{12} + S_{11} = (B_{21} + S_{22}) = \lambda \\ B_{22} &= (2\mu + \lambda)(1 + \epsilon_{22} - \epsilon_{11}); \quad Q = \mu + 1/2(\lambda + \mu)(\epsilon_{11} + \epsilon_{22}). \end{aligned} \tag{3}$$

using (1) in (3) one gets

$$\begin{aligned} S_{11} &= (2\mu + \lambda)\{1 - (S_{22} - S_{11})/2\mu\} \\ B_{12} &= \lambda - S_{11} \\ B_{22} &= (2\mu + \lambda)\{1 - (S_{11} - S_{22})/2\mu\} \\ B_{21} &= \lambda - S_{22} \\ \text{and} \quad Q &= \mu + \frac{(S_{11} + S_{22})(\lambda + \mu)(\lambda + 2\mu)}{4\mu(3\lambda + 2\mu)}. \end{aligned} \tag{4}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The above result shows that the presence of initial stress modifies the elastic constants.

Particular Cases. If the medium has initial stress S_{11} only along x_1 direction and $S_{22} = 0$ then the incremental elastic coefficients are

$$B_{11} = (2\mu + \lambda)\left(1 + \frac{S_{11}}{2\mu}\right); \quad B_{12} = \lambda - S_{11};$$

$$B_{22} = (2\mu + \lambda)\left(1 - \frac{S_{11}}{2\mu}\right); \quad B_{21} = \lambda$$

$$\text{and} \quad Q = \mu + \frac{S_{11}(\lambda + \mu)(\lambda + 2\mu)}{4\mu(3\lambda + 2\mu)}. \tag{5}$$

Equation (5) shows that, under tensile initial stresses, B_{11} and Q increase as the initial stress increases, however, B_{21} is not affected by the presence of S_{11} along x_1 direction, whereas B_{22} and B_{12} decrease. In the case of initial compressive stress $P = -S_{11}$ along x_1 direction the incremental elastic coefficients are

$$B_{11} = (2\mu + \lambda)\left(1 - \frac{P}{2\mu}\right); \quad B_{12} = \lambda + P; \quad B_{22} = (2\mu + \lambda)\left(1 + \frac{P}{2\mu}\right)$$

$$Q = \mu - \frac{P(\lambda + \mu)(\lambda + 2\mu)}{4\mu(3\lambda + 2\mu)}; \quad B_{21} = \lambda \tag{6}$$

which may be expressed in the non-dimensional form as

$$\frac{B_{11}}{\mu} = \left(2 + \frac{\lambda}{\mu}\right)(1 - \zeta); \quad \frac{B_{12}}{\mu} = \left(\frac{\lambda}{\mu} + 2\zeta\right); \quad \frac{B_{22}}{\mu} = \left(2 + \frac{\lambda}{\mu}\right)(1 + \zeta)$$

$$\frac{Q}{\mu} = 1 - \frac{\zeta\left(\frac{\lambda}{\mu} + 1\right)\left(\frac{\lambda}{\mu} + 2\right)}{2\left(3\frac{\lambda}{\mu} + 2\right)}; \quad \frac{B_{21}}{\mu} = \frac{\lambda}{\mu} \tag{7}$$

where $\zeta = \frac{P}{2\mu}$ is the initial stress parameter and it is positive or negative according as the initial stresses are compressive and tensile, respectively. Hence, B_{11} and Q decrease whereas B_{22} and B_{12} increase as the initial compressive stress increase.

From (7) it may be concluded that the presence of compressive stress along u_1 direction reduces the rigidity of the medium and the material becomes softer as the compressive stress increases. The rigidity of the material will be completely zero at the compressive stress

$$\zeta = \frac{2(2 + 3\lambda/\mu)}{(1 + \lambda/\mu)(2 + \lambda/\mu)} \tag{8}$$

and hence the elastic medium will be changed to fluid when the medium is subjected to compressive stress given in (8).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Numerical Calculation. The numercial result of Q/μ the ratio of effective rigidity of the medium with initial stress have been calculated for different values of $\zeta = P/2\mu$ and for different values of λ/μ and the results are presenting in Fig. (1). The Figure (2) shows that the variation of other elastic constant B_{22}/μ with respect to ζ and for different values of λ/μ.

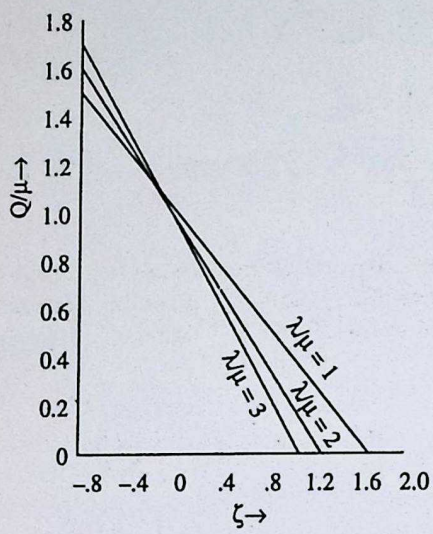


Fig 1 Variation of Q/μ with respect to ζ for different values of λ/μ

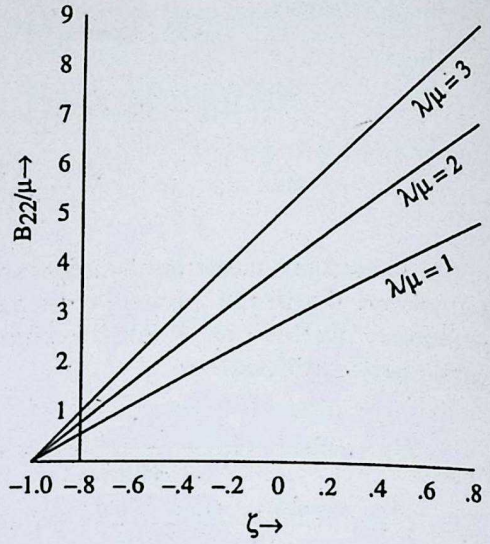


Fig 2 Variation of B_{22}/μ with respect to ζ for different values of λ/μ

2. Rayleigh Wave in an Initially Stressed Half Space Boundary by a Rigid Layer.

Formulation. Let a homogeneous isotropic half space be subjected to initial compressive stress $S_{11} = -P$ along x_1 direction. Let x_2 direction be pointing upward and the boundary of the half space is $x_2 = 0$. The half space is bounded by rigid layer at the top Fig. (3).

The dynamical equations of motion are

$$\frac{\partial s_{11}}{\partial x_1} + \frac{\partial s_{12}}{\partial x_2} - P\frac{\partial w}{\partial x_2} = \rho\frac{\partial^2 u}{\partial t^2};$$

$$\frac{\partial s_{12}}{\partial x_1} + \frac{\partial s_{22}}{\partial x_2} - P\frac{\partial w}{\partial x_1} = \rho\frac{\partial^2 v}{\partial t^2}, \quad (9)$$

where s_{ij} are incremental stress coneffficients.

$$w = \frac{1}{2}\left(\frac{\partial v}{\partial x_1} - \frac{\partial u}{\partial x_2}\right),$$

the rotational components.

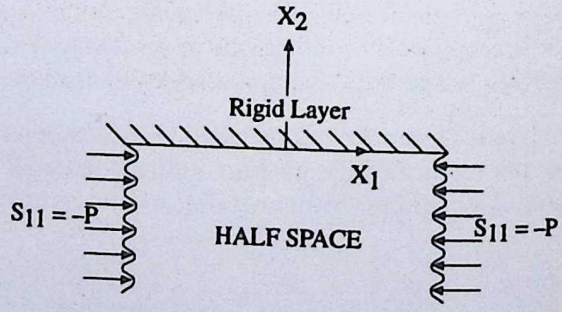


Fig 3 Model of Rayleigh Wave

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

u, v are displacement components along x_1 and x_2 directions, respectively, ρ is the nsity of the half space.

The stress-strain relations are given in equation (2) of Section I and B_{ij} are incremental stic coefficients whose value have been discussed in Section I and in the present case they e given by

$$B_{11} = (2\mu + \lambda)\left(1 - \frac{P}{2\mu}\right); \quad B_{22} = (2\mu + \lambda)\left(1 + \frac{P}{2\mu}\right) \tag{10}$$

$$B_{12} = (\lambda + P); \quad Q = \mu - \frac{P(\lambda + \mu)(\lambda + 2\mu)}{4\mu(3\lambda + 2\mu)}; \quad B_{12} = B_{21} + P$$

here, λ and μ are the elastic coefficients of isotropic initial stress-free medium.

sing (2) in (9) one gets

$$B_{11}\frac{\partial^2 u}{\partial x_1^2} + \left(Q + \frac{P}{2}\right)\frac{\partial^2 u}{\partial x_2^2} + \left(B_{12} + Q - \frac{P}{2}\right)\frac{\partial^2 v}{\partial x_1 \partial x_2} = \rho\frac{\partial^2 u}{\partial t^2}$$

$$B_{22}\frac{\partial^2 v}{\partial x_2^2} + \left(Q - \frac{P}{2}\right)\frac{\partial^2 v}{\partial x_1^2} + \left(B_{21} + Q + \frac{P}{2}\right)\frac{\partial^2 u}{\partial x_1 \partial x_2} = \rho\frac{\partial^2 v}{\partial t^2} \tag{11}$$

or the sinusoidal wave along x_1 direction the solution (11) may be taken as

$$u = U(kx_2)e^{i(kx_1 - pt)}; \quad v = V(kx_2)e^{i(kx_1 - pt)}. \tag{12}$$

Jsing (11) the equations in (10) take the form

$$B_{11}U(kx_2) - \left(Q + \frac{P}{2}\right)U''(kx_2) + \left(B_{12} + Q - \frac{P}{2}\right)V'(kx_2) - \left(\frac{P^2}{k^2}\right)\rho U(kx_2) = 0$$

$$B_{22}V''(kx_2)\left(Q - \frac{P}{2}\right)V(kx_2) + \left(B_{21} + Q + \frac{P}{2}\right)U'(kx_2) + \frac{P^2}{k^2}\rho V(kx_2) = 0. \tag{13}$$

Eliminating $U(kx_2)$and $V(kx_2)$, successively from (13) and using relation and utilizing the last equation of (10):

The following ordinary differential equation for $U(kx_2)$ and $V(kx_2)$ may be obtained

$$U'''' - \left(\frac{1}{B_{22}L}\right)\left[B_{22}R - L\left(2B_{21} + P + \frac{P^2}{k^2}\rho\right) - B_{21}^2\right]$$

$$U'' + \frac{R}{B_{22}L}\left(L - P - \frac{P^2}{k^2}\rho\right)U = 0$$

$$V'''' - \frac{1}{B_{22}L}\left[B_{22}R - L\left(2B_{21} + P - \frac{P^2}{k^2}\rho\right) - B_{21}^2\right]$$

$$V'' + \frac{R}{B_{22}L}\left(L - P - \frac{P^2}{k^2}\rho\right)V = 0, \tag{14}$$

where, $L = Q + \frac{P}{2}R = B_{11} - \frac{P^2}{k^2}\rho$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

For the Rayleigh wave propagating along x_1 direction solution of (14) may be taken as

$$U(kx_2) = C_1 e^{\beta_1 k x_2} + C_2 e^{\beta_2 k x_2}; \quad V(kx_2) = C_1' e^{\beta_1 k x_2} + C_2' e^{\beta_2 k x_2} \tag{15}$$

where, β_1 and β_2 are the roots of the characteristic equation

$$\beta^4 - 2m\beta^2 + s^2 = 0 \tag{16}$$

with

$$2m = \frac{1}{LB_{22}}\left[\Delta B_{22} - L\left(2B_{21} + P + \frac{P^2}{k^2}\rho\right) - B_{21}^2\right]$$
$$s^2 = \Delta / LB_{22}\left(L - P - \frac{P^2}{k^2}\rho\right)$$

where, $\Delta = B_{11} - \frac{P^2}{k^2}\rho$.

The values of β_1^2 and β_2^2 are

$$\beta_1^2 = m + \sqrt{(m^2 - s^2)}, \beta_2^2 = m - \sqrt{(m^2 - s^2)}. \tag{17}$$

The signs of β_1 and β_2 may be so chosen that they possess no negative real part. The arbitrary constants C_i and C_i' are not independent.

Taking

$$U = C_i e^{k x_2 \beta_i}; \quad V = C_i' e^{k x_2 \beta_i}$$

and substituting in any one of (13) one can find out the relationship between C_i and C_i' as

$$C_i' = \frac{-\Delta + L\beta_1^2}{(B_{21} + L)\beta_i} C_i. \tag{18}$$

Hence the solution of (11) are

$$u = \left(C_1 e^{\beta_1 k x_2} + C_2 e^{\beta_2 k x_2}\right) e^{i(kx_1 - pt)}; \quad v = \left(C_1' e^{\beta_1 k x_2} + C_2' e^{\beta_2 k x_2}\right) e^{i(kx_2 - pi)} \tag{19}$$

where, C_i and C_i' are connected by (18).

Boundary Conditions. As the boundary $x_2 = 0$ is covered by a rigid layer the boundary conditions are

$$u = 0, v = 0 \quad \text{at} \quad x_2 = 0 \tag{20}$$

using (19) the conditions (20) take the form

$$C_1 + C_2 = 0; \quad \frac{-\Delta + L\beta_1^2}{(B_{21} + L)\beta_1} C_1 + \frac{-\Delta + L\beta_2^2}{(B_{21} + L)\beta_2} C_2 = 0. \tag{21}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

r non-trivial solution of C_1 and C_2 one must have

$$\begin{vmatrix} 1 & 1 \\ \dfrac{-\Delta + L\beta_1^2}{(B_{21}+L)\beta_1} & \dfrac{-\Delta + L\beta_2^2}{(B_{21}+L)\beta_2} \end{vmatrix} = 0$$

hich gives $(-\Delta + L\beta_2^2)\beta_1 - (-\Delta + L\beta_1^2)\beta_2 = 0$.

utting the values of Δ and L and on simplification

$$(\beta_1 - \beta_2)\left[\frac{B_{11}}{Q} - \frac{c^2\rho}{Q} + \left(1 + \frac{P}{2Q}\right)\beta_1\beta_2\right] = 0 \tag{22}$$

hich gives either $\beta_1 = \beta_2$ (23)

$$\text{or,} \quad \frac{B_{11}}{Q} - \frac{c^2\rho}{Q} + \left(1 + \frac{P}{2Q}\right)\beta_1\beta_2 = 0. \tag{24}$$

aking (24) and putting the value of $\beta_1\beta_2$, one gets

$$\frac{B_{11}}{Q} - \frac{c^2}{Q} + \sqrt{\left[\left(1 + \frac{P}{2Q}\right)\left(\frac{B_{11} - c^2\rho}{B_{22}}\right)\left(1 - \frac{P}{2Q} - \frac{c^2\rho}{Q}\right)\right]} = 0, \tag{25}$$

hich on simplification may be expressed in non-dimensionalised form as

$$\left(\frac{c^2}{Q/\rho}\right)^2\left[1 - \frac{Q}{B_{22}}\left(1 + \frac{P}{2Q}\right)\right] - \left(\frac{c^2}{Q/\rho}\right)\left[2\frac{B_{11}}{Q} - \frac{Q}{B_{22}} + \frac{P^2}{4Q^2}\frac{Q}{B_{22}} - \left(1 + \frac{P}{2Q}\right)\frac{B_{11}}{B_{22}}\right]$$
$$+ \frac{B_{11}^2}{Q^2} - \frac{B_{11}}{B_{22}} + \frac{P^2}{4Q^2}\cdot\frac{B_{11}}{B_{22}} = 0. \tag{26}$$

Equation (26) gives the velocities of Rayleigh wave propagating in the half space.

Considering the case in (22) that is $\beta_1 = \beta_2$.

The solution (19) takes the form

$$u = (C_1 + C_2 k x_2)e^{\beta k x_2}e^{i(kx_1 - pt)}; \quad v = (C_1' + C_2' k x_2)e^{\beta k x_2}e^{i(kx_1 - pt)} \tag{27}$$

and the boundary condition (20) gives $C_1 = 0$ and $C_1' = 0$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Solution (27) takes the form

$$u = C_2 e^{\beta k x_2} e^{i(kx_1 - pt)}; \; v = C_2' = e^{\beta \bar{k} x_2} e^{i(kx_1 - pt)} \tag{28}$$

which gives the displacement ratio as

$$\frac{v}{u} = \frac{-\Delta + Lm}{(B_{21} + L)\sqrt{m}} = \frac{-\left(B_{11} - \frac{P^2}{k^2}\rho\right) + \left(Q + \frac{P}{2}\right)m}{\left(B_{21} + Q + \frac{P}{2}\right)\sqrt{m}}. \tag{29}$$

For $\beta_1 = \beta_2$, from the discriminant of equation (16) one gets $m^2 = s^2$ which on calculation gives the velocity equation of the Rayleigh wave as

$$A_1\left(\frac{C_R^4}{C_T^4}\right) + B_1\left(\frac{C_R^2}{C_T^2}\right) + C_1 = 0, \tag{30}$$

where

$$A_1 = 9(1+2\zeta)^2 + \frac{1}{4}\left(1 + \frac{2}{5}\zeta\right)^2 - 9(1+\zeta)\left(1 + \frac{2}{5}\zeta\right)$$

$$B_1 = 12(1+\zeta)\left(1 + \frac{2}{5}\zeta\right)\left(5 - \frac{11}{5}\zeta\right) + \left(1 + \frac{2}{5}\zeta\right) + 6(1+\zeta)(9\zeta^2 - 8)$$

$$C_1 = (1-\zeta^2)\left(1 + \frac{2}{5}\zeta\right)\left(\frac{198}{5}\zeta - 54\right) + 9(1-\zeta^2)(7 - 9\zeta^2)$$
$$+ (1+\zeta)\left(1 + \frac{2}{5}\zeta\right)\left(4 + \frac{7}{5}\zeta + \frac{2}{5}\zeta^2\right) + 1.$$

Therefore, the velocity equation giving the velocity of Rayleigh waves in an initially stressed half space under rigid boundary are given by equations (26) and (30) and hence there should be at least two wave fronts propagating with different velocities.

Particular Case. In case the medium is free from initial stresses, that is $\zeta = 0$.

Then $Q = \mu, B_{11} = +2\mu$

$$\beta_1 = \sqrt{(1 - C_R^2/\alpha^2)} \quad \text{and} \quad \beta_2 = \sqrt{(1 - C_R^2/\beta^2)}$$

and the equation (24) takes the form

$$\sqrt{\left(1 - \frac{C^2R}{\alpha^2}\right)}\left[\frac{\alpha^2}{\beta^2}\sqrt{\left(1 - \frac{C_R^2}{\alpha^2}\right)} + \sqrt{\left(1 - \frac{C_R^2}{\beta^2}\right)}\right] = 0 \tag{31}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The first factor of equation(31) gives $c = \alpha$ and the second factor gives either $\alpha = \beta$ $c^2 = \alpha^2 + \beta^2$; both the latter conditions are not genuine and hence the only solution of (31) $c = \alpha$. The condition $\beta_1 = \beta_2$ will give only $c = 0$.

Therefore, it may be concluded that the velocity under a rigid boundary and without itial stress is equal to the velocity of longitudinal wave in the medium. It is known that velocity Rayleigh waves (C_R) in a half space with free boundary satisfies the relation $C_R < \beta < \alpha$.

Hence, the presence of rigid layer over the half space increases the velocity of Rayleigh aves.

Numerical Calculation. The numerical values of $\left(\frac{C_R}{C_T}\right)^2$ have been calculated from quation (26) for different values of λ/μ and for $\zeta = -.4$ as well as, for $\zeta = -.8$. The results ave been presented in Fig. (4). The curves show that as λ/μ increases the values of $\left(\frac{C_R}{C_T}\right)^2$ also ncreases but the increase is more rapid for low values of ζ than the high values. It may also e marked that as ζ increases the wave starts its appearance at higher degree of anisotropy.

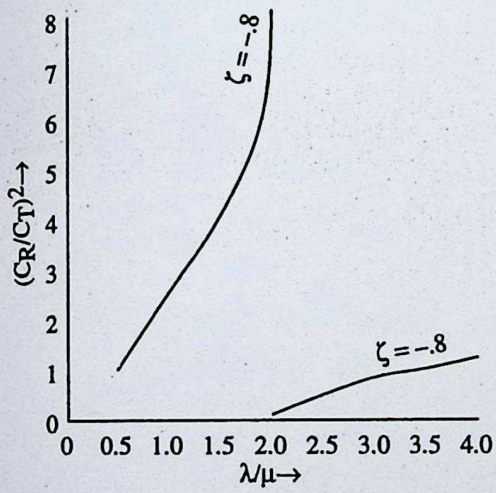


Fig 4 Variation of $(C_R/C_T)^2$ with respect to λ/μ for $\zeta = -.4$ and $\zeta = -.8$

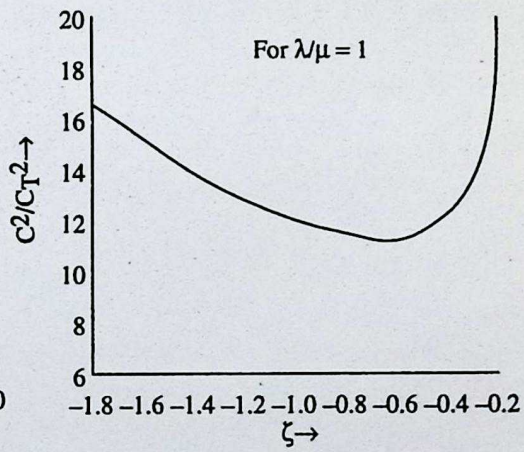


Fig 5 Variation of C^2/C_T^2 with respect to 5

The numerical values of $\left(\frac{C_R}{C_T}\right)^2$ have also been calculated from equation (30) at $\lambda = \mu$ for different values of ζ and it was observed that there can be two Rayleigh wave fronts which exists for different range of values of ζ. The velocities of both the fronts have been shown in Fig. (5). The continuous curve represents velocities the first front of which exists for $\zeta = -1.8$ to $-.6$ and the doted curve represents the velocities for the second front which starts after $\zeta = -.6$ and exists up to ζ nearer to zero and disappear at $\zeta = 0$. Thus, the study reveals that Rayleigh wave may propagate in the half space bounded by a rigid layer only when the half space is under initial stress and does not propagate when the medium is free from initial stress.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Conclusion. The presence of rigid boundary over a half space has effect on the propagation of Rayleigh wave. If the medium is free from initial stresses, the wave does not propagate. The presence of initial stresses makes the wave propagate sometimes in one front and sometimes in two fronts, depending upon the values of initial stress parameter. It may be marked that in the absence of rigid boundary at the top, the half space allows only one Rayleigh wave front in presence or in the absence of initial stresses, both.

* PROFESSOR OF MATHEMATICS
INDIAN SCHOOL OF MINES
DHANBAD 826 004, INDIA

** LECTURER IN MATHEMATICS
R. S. MORE COLLEGE
GOBINDAPUR
DHANBAD 828 109

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

ull. Cal. Math. Soc., 86, 547–554 (1994)

EIGENFUNCTION EXPANSIONS FOR A REGULAR RIGHT-DEFINITE EIGENVALUE PROBLEM OF ORDER 2N WITH AN EIGENPARAMETER IN THE BOUNDARY CONDITIONS

E. M. E. ZAYED AND S. F. M. IBRAHIM

(*Received 21 August 1993*)

Abstract. The object of this paper is to establish an expansion theorem for a regular ight-definite eigenvalue problem of order $2n$ with an eigenvalue parameter λ in the boundary onditions. We associate with this problem a self-adjoint operator with compact resolvent efined in a suitable Hilbert space and then we develop an associated eigenfunction expansion heorem.

1. Introduction. The regular right-definite eigenvalue problems for second and ourth orders ordinary differential equations with an eigenvalue parameter in the boundary onditions have been studied by Walter (1973), Schneider (1974), Fulton (1977), Hinton (1979), Zayed and Ibrahim (1989), Zayed (1984) and the references given there.

The object of this paper is to study the following regular right-definite eigenvalue problem of order $2n$ consisting of the ordinary differential equation

$$\tau y(x) := \sum_{i=0}^{n} (-1)^{n-i} \left\{ P_i(x) y^{(n-i)}(x) \right\}^{(n-i)} = \lambda y(x), \text{ for } x \in [a,b], \tag{1.1}$$

together with the boundary conditions

$$\left.\begin{aligned} & y^{(i)}(a) = 0, \quad (i = 0, 1, \ldots, n-1), \\ & \left\{ P_i(x) y^{(n-i)}(x) \right\}^{(n-i-2)}(b) = 0, \quad (i = 0, 1, \ldots, n-2), \\ & \beta y(b) := \sum_{i=0}^{n-1} (-1)^{n-i} \left\{ P_i(x) y_{(x)}^{(n-i)} \right\}^{(n-i-1)}(b) = -\lambda y(b), \end{aligned}\right\} \tag{1.2}$$

where $P_i(x) > 0, (i = 0, 1, \ldots, n)$ are real-valued continuous functions on $[a,b]$, while λ is a complex number and $y^{(n)}(x) = \frac{d^n y(x)}{dx^n}$.

Our approach in this paper is to find a suitable Hilbert space H and a self-adjoint operator A with compact resolvent defined in H in such a way that the problem (1.1)–(1.2) can be considered as an eigenvalue problem of the operator A.

We close this section with the remark that the problem (1.1)–(1.2) can be considered as a more generalisation of those obtained in Zayed and Ibrahim (1989) and Zayed (1984).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

2. Hilbert space formulation

DEFINITION 1. *We define a Hilbert space H of two component vectors by*

$$H = L^2[a,b] \oplus C; \tag{2.1}$$

with the inner product

$$< f(x), g(x) >_H = \int_a^b f_1(x)\bar{g}_1(x)dx + f_2\bar{g}_2, \tag{2.2}$$

and the norm

$$\left\| f(x) \right\|_H^2 = \int_a^b \left| f_1(x) \right|^2 dx + \left| f_2 \right|^2, \tag{2.3}$$

for all $f(x) = (f_1(x), f_2)$ and $g(x) = (g_1(x), g_2)$ in H.

DEFINITION 2. *We define a linear operator $A : D(A) \to H$ by*

$$Af(x) = (\tau f_1(x), -\beta f_1(b)), \forall f(x) \in D(A), \tag{2.4}$$

Where the domain $D(A)$ of A is defined as the set of all $f(x) = (f_1(x), f_2) \in H$, which satisfies the following conditions:

(i) $f_1^{(2n-i)}(x)$, $(i = 1, 2, \ldots, 2n)$, *are absolutely continuous functions on* $[a,b]$ *with* $\tau f_1 \in L^2[a,b]$,

(ii) $f_1^{(i)}(a) = 0, (i = 0, 1, \ldots, n-1)$,

(iii) $\left\{ P_i(x) f_1^{(n-i)}(x) \right\}^{(n-i-2)}(b) = 0, (i = 0, 1, \ldots, n-2)$,

(iv) $f_2 = f_1(b)$.

REMARK 1. λ is an eigenvalue and $f_1(x)$ is a corresponding eigenfunction of the problem (1.1)–(1.2) if, and only if, $f(x) = (f_1(x), f_1(b)) \in D(A)$ and $Af(x) = \lambda f(x)$. Therefore, the eigenvalues and the eigenfunctions of the problem (1.1)–(1.2) are equivalent to the eigenvalues and the eigenfunctions of the linear operator A.

REMARK 2. $D(A)$ is dense in H.

LEMMA 1. *The linear operator A in H is symmetric.*

Proof. Let $f(x) = (f_1(x), f_1(b))$ and $g(x) = (g_1(x), g_1(b))$ be any two elements in $D(A)$. Then, we write

$$\begin{aligned} < Af(x), g(x) >_H &= \int_a^b (\tau f_1(x))\bar{g}_1(x)dx - (\beta f_1(b)\bar{g}_1(b) \\ &= \int_a^b \sum_{i=0}^{n} (-1)^{(n-i)} \left\{ P_i(x) f_1^{(n-i)}(x) \right\}^{(n-i)} \bar{g}_1(x)dx \\ &\quad - \sum_{i=0}^{n-1} (-1)^{n-i} \left\{ P_i(x) f_1^{(n-i)}(x) \right\}^{(n-i-1)}(b)\bar{g}_1(b). \end{aligned}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

On integrating by parts and using the boundary conditions, we deduce after some reduction that

$$
\begin{aligned}
< Af(x), g(x) >_H &= \int_a^b f_1(x) \sum_{i=0}^{n} (-1)^{n-i} \left\{ P_i(x) \bar{g}_1^{(n-i)}(x) \right\}^{(n-i)} dx \\
&\quad - f_1(b) \sum_{i=0}^{n-1} (-1)^{n-i} \left\{ P_i(x) \bar{g}_1^{(n-i)}(x) \right\}^{(n-i-1)} (b) \\
&= \int_a^b f_1(x)(\tau \bar{g}_1(x) dx - f_1(b)(\beta \bar{g}_1(b)) \\
&= < f(x), Ag(x) >_H \qquad (2.5)
\end{aligned}
$$

This proves that A in H is symmetric.

REMARK 3. Since the linear operator A in H is symmetric, it has only real eigenvalues.

LEMMA 2. *If $f_1(x)$ is an absolutely continuous function with $2(n-i)$ continuous derivatives, $(i = 0, 1, \ldots, n-2)$ on $[a, b]$ with $f_1(a) = 0$. Since $P_i(x) > 0, (i = 0, 1, \ldots, n)$ are real-valued continuous functions on $[a, b]$, then there exists real positive constants C_i with $P_i(x) \geq C_i (i = 0, 1, \ldots, n)$ $\forall x \in [a, b]$ such that*

$$\int_a^b P_n(x) \left| f_1(x) \right|^2 dx \geq C_n \int_a^b \left| f_1(x) \right|^2 dx, \qquad (2.6)$$

$$\int_a^b P_{n-1}(x) \left| f_1^{(1)}(x) \right|^2 dx \geq \frac{C_{n-1}}{(b-a)} \left| f_1(b) \right|^2, \qquad (2.7)$$

and
$$\int_a^b \sum_{i=0}^{n-2} P_i(x) \left| f_1^{(n-i)}(x) \right|^2 dx \geq \sum_{i=0}^{n-2} \frac{C_i}{\left[16(b-a)^2\right]^{n-i}} \int_a^b \left| f_1(x) \right|^2 dx. \qquad (2.8)$$

Proof. The proof of (2.6) is obvious. To prove (2.7) we use Schwartz's inequality, to give

$$
\begin{aligned}
\int_a^b P_{n-1}(x) \left| f_1^{(1)} \right|^2 dx &\geq C_{n-1} \int_a^b \left| f_1^{(1)}(x) \right|^2 dx \geq \frac{C_{n-1}}{(b-a)} \left| \int_a^b f_1^{(1)}(x) dx \right|^2 \\
&= \frac{C_{n-1}}{(b-a)} \left| f_1(b) \right|^2
\end{aligned}
$$

To prove (2.8) we use Theorem 2 in (Hellwig 1967, P. 67), to get for $f_1 \in G^1[a, b]$

$$\int_a^b \left| f_1(x) \right|^2 dx \leq 4(b-a)^2 \int_a^b \left| \quad \left| f_1(x) \right|^{(1)} \right|^2 dx. \qquad (2.9)$$

Since

$$\int_a^b \left| \left| f_1(x) \right|^{(1)} \right|^2 dx \leq 4 \int_a^b \left| f_1^{(1)}(x) \right|^2 dx,$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

we obtain

$$\int_a^b \left|f_1(x)\right|^2 dx \leq 16(b-a)^2 \int_a^b \left|f_1^{(1)}(x)\right|^2 dx. \tag{2.10}$$

Applying (2.10) again for $\left|f_1^{(1)}(x)\right.$ we get

$$\int_a^b \left|f_1^{(1)}(x)\right|^2 dx \leq 16(b-a)^2 \int_a^b \left|f_1^{(2)}(x)\right|^2 dx \tag{2.11}$$

From (2.10)–(2.11) we obtain

$$\int_a^b \left|f_1(x)\right|^2 dx \leq \left[16(b-a)^2\right]^2 \int_a^b \left|f_1^{(2)}(x)\right|^2 dx \tag{2.12}$$

Repetition of these steps leads us inductively to have the inequality

$$\int_a^b \left|f_1(x)\right|^2 dx \leq \left[16(b-a)^2\right]^n \int_a^b \left|f_1^{(n)}(x)\right|^2 dx. \tag{2.13}$$

With the help of (2.13), we get

$$\begin{aligned}\int_a^b \sum_{i=0}^{n-2} P_i(x)\left|f_1^{(n-i)}(x)\right|^2 dx &\geq \int_a^b \sum_{i=0}^{n-2} C_i\left|f_1^{(n-i)}(x)\right|^2 dx \\ &\geq \sum_{i=0}^{n-2} \frac{C_i}{\left[16(b-a)^2\right]^{n-i}} \int_a^b \left|f_1(x)\right|^2 dx,\end{aligned} \tag{2.14}$$

and then the proof of Lemma 2 follows.

LEMMA 3. *The linear operator A in H is bounded below and strictly positive.*

Proof. Let $f(x) = (f_1(x), f_1(b))$ be any element in $D(A)$. Then

$$\begin{aligned}< Af(x), f(x) >_H = &\int_a^b \sum_{i=0}^{n} (-1)^{n-i}\left\{P_i(x) f_1^{(n-i)}(x)\right\}^{(n-i)} \bar{f}_1(x)dx \\ &- \sum_{i=0}^{n-1} (-1)^{n-i}\left\{P_i(x) f_1^{(n-i)}(x)\right\}^{(n-i-1)}(b)\bar{f}_1(b).\end{aligned}$$

On integrating by parts and using the boundary conditions, we get

$$< Af(x), f(x) >_H = \int_a^b \sum_{i=0}^{n} P_i(x)\left|f_1^{(n-i)}(x)\right|^2 dx. \tag{2.15}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

On inserting (2.6)–(2.8) into (2.15), we deduce that

$$< Af(x), f(x) >_H \geq \left\{ \sum_{i=0}^{n-2} \frac{C_i}{\left[16(b-a)^2\right]^{n-i}} + C_n \right\} \int_a^b \left|f_1(x)\right|^2 dx + \frac{C_{n-1}}{(b-a)} \left|f_1(b)\right|^2. \tag{2.16}$$

Define a real number γ by

$$\gamma = \min \left\{ \sum_{i=0}^{n-2} \frac{C_i}{\left[16(b-a)^2\right]^{n-i}} + C_n, \frac{C_{n-1}}{(b-a)} \right\}. \tag{2.17}$$

From (2.16)–(2.17) we find that

$$< Af(x), f(x) >_H \geq \gamma \left\| f(x) \right\|_H^2 \tag{2.18}$$

It now follows from (2.18) that A in H is bounded below.

Since $C_i (i = 0, 1, \ldots, n)$ are positive constants, then $\gamma > 0$. This proves that A in H is strictly positive.

LEMMA 4. *The linear operator A in H is unbounded above.*

Proof. Let $\phi(x)$ be a test function with compact support on $[a, b]$. Define a sequence of this test function with compact support on $[a, b]$ as follows:

$$\phi_m(x) = \phi(mx) \quad \forall x \in [a, b], m = 1, 2, \ldots.$$

On using arguments similar to those used in Lemma 3, we can show that

$$< A\phi_m(x), \phi_m(x) >_H \geq \left\{ \sum_{i=0}^{n-2} C_i \left[\frac{m^2}{16(b-a)^2} \right]^{n-i} + C_n \right\} \left\| \phi_m(x) \right\|_H^2. \tag{2.19}$$

Letting $m \to \infty$ in (2.19), we get

$$\lim_{m \to \infty} < A\phi_m(x), \phi_m(x) >_H = \infty. \tag{2.20}$$

This proves that A in H is unbounded above.

REMARK 4. (i) Since A in H is bounded below, then the set of all eigenvalues of A is also bounded below by the constant γ defined by (2.17).

(ii) Since A in H is strictly positive, then $\lambda = 0$ is not an eigenvalue of A.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

3. The resolvent operator and the expansion theorem.

We denote by $Y_\nu = Y_\nu(x;\lambda), \nu = 1,2,\ldots,2n$, a system of solutions of the equation $\tau y = \lambda y$ on $[a,b]$ which satisfy the initial conditions:

$$\lim_{x\to a} +Y_\nu(x;\lambda) = \lim_{x\to a} +[P_{n-1}(x)Y_\nu^{(1)}(x;\lambda)] = 0, \quad (\nu = 1,\ldots,n), \tag{3.1}$$

$$\lim_{x\to a} +\left\{P_i(x)Y_\nu^{(n-i)}(x;\lambda)\right\}^{(n-i-2)} = \begin{cases} 1 & \text{if } \nu = 1,\ldots,n-1. \\ o & \text{if } \nu = n, \end{cases} \tag{3.2}$$

where $i = 0,1,\ldots,n-2$,

$$\lim_{x\to a} +\sum_{i=0}^{n-2}(-1)^{(n-i)}\left\{P_i(x)Y_\nu^{(n-i)}(x;\lambda)\right\}^{(n-i-1)} = \begin{cases} 0 & \text{if } \nu = 1,2,\ldots,n-1 \\ 1 & \text{if } \nu = n, \end{cases} \tag{3.3}$$

and

$$\lim_{x\to b} -Y_\nu(x;\lambda) = \begin{cases} 0 & \text{if } \nu = n+1,\ldots,2n-1 \\ 1 & \text{if } \nu = 2n, \end{cases} \tag{3.4}$$

$$\lim_{x\to b} -[P_{n-1}(x)Y_\nu^{(1)}(x;\lambda)] = \begin{cases} 1 & \text{if } \nu = n+1,\ldots,2n-1 \\ 1+\lambda & \text{if } \nu = 2n, \end{cases} \tag{3.5}$$

$$\lim_{x\to b} -\left\{P_i(x)Y_\nu^{(n-i)}(x;\lambda)\right\}^{(n-i-2)} = 0 \quad \text{if} \quad \nu = n+1,\ldots,2n, i = 0,1,\ldots n-2, \tag{3.6}$$

$$\lim_{x\to b} -\sum_{i=0}^{n-2}(-1)^{n-i}\left\{P_i(x)Y_\nu^{(n-i)}(x;\lambda)\right\}^{(n-i-1)} = 1 \quad \text{if} \quad \nu = n+1,\ldots,2n. \tag{3.7}$$

We take the initial conditions (3.1)–(3.7) in order to satisfy that the Wronskian $W(Y_1, \ldots, Y_{2n})$ is different from zero. Therefore, $Y_\nu(x;\lambda), \nu = 1,\ldots,2n$ are linearly independent solutions of the O.D.E. (1.1). Now, for $f(x) = (f_1(x), f_2) \in H$ we define $\Phi(x) = (\Phi_1(x), \Phi_2) \in D(A)$ as the unique solution of the equation $(\lambda I - A)\Phi(x) = f(x)$.

Therefore,

$$(\lambda I - \tau)\Phi_1(x) = f_1(x), \tag{3.8}$$

$$\text{and} \quad \lambda\Phi_1(b) + \sum_{i=0}^{n-1}(-1)^{n-i}\left\{P_i(x)\Phi_1^{(n-i)}(x)\right\}^{(n-i-1)}(b) = f_2. \tag{3.9}$$

On applying the method of variation of parameters, we get

$$\Phi_1(x) = \int_a^b \sum_{\nu=1}^{n}\left(\frac{Y_\nu(x;\lambda)K_\nu(t)}{W}\right)f_1(t)dt + 2\int_a^b \sum_{\nu=n+1}^{2n} \left(\frac{Y_\nu(x;\lambda)K_\nu(t)}{W}\right)f_1(t)dt + \sum_{\nu=1}^{2n} d_\nu Y_\nu(x;\lambda), \tag{3.10}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Where $d_\nu(\nu = 1, \ldots, 2n)$ are arbitrary constants to be determined, while $K_\nu(t) = C_\nu(t)$, and $C_\nu(t)$ can be written in the form:

$$C_\nu(t) = (-1)^\nu \frac{P_{n-1}(t)}{P_{n-2}(t)} \begin{vmatrix} Y_1(t) & Y_2(t) & \ldots & Y_{\nu-1} & Y_{\nu+1}(t) & \ldots & Y_{2n}(t) \\ Y_1^{(1)}(t) & Y_2^{(1)}(t) & \ldots & Y_{\nu-1}^{(1)}(t) & Y_{\nu+1}^{(1)}(t) & \ldots & Y_{2n}^{(1)}(t) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ Y_1^{(2n-2)}(t) & Y_2^{(2n-2)}(t) & \ldots & Y_{\nu-1}^{(2n-2)}(t) & Y_{\nu+1}^{(2n-2)}(t) & \ldots & Y_{2n}^{(2n-2)}(t) \end{vmatrix}, \quad (3.11)$$

Where $\nu = 1, 2, \ldots, 2n$.

From (3.9), (3.10) together with the initial conditions (3.1)–(3.7) we get

$$d_\nu = \begin{cases} \dfrac{1}{W}\left[(-1)^\nu f_2\left\{P_o(x)Y_{n+1-\nu}^{(n)}(x;\lambda)\right\}^{(n-2)}(b) + \displaystyle\int_a^b K_\nu(t)f_1(t)dt\right] & \text{if } \nu = 1, \ldots, n \\ 0 & \text{if } \nu = n+1, \ldots, 2n. \end{cases} \quad (3.12)$$

Consequently, we find that

$$\Phi_1(x) = \frac{f_2}{W}\sum_{\nu=1}^{n}(-1)^\nu\left\{P_o(x)Y_{n+1-\nu}^{(n)}(x;\lambda)\right\}^{(n-2)}(b)Y_\nu(x;\lambda) + \int_a^b G(x,t;\lambda)f_1(t)dt, \quad (3.13)$$

and

$$\Phi_2 = \Phi_1(b), \quad (3.14)$$

Where $G(x,t;\lambda)$ is the Green's function of the problem (1.1)–(1.2) defined by

$$G(x,t;\lambda) = \begin{cases} \displaystyle\sum_{\nu=1}^{n}\frac{Y_\nu(x;\lambda)C_\nu(t)}{W} & \text{if } a \le x \le t \le b, \\ \displaystyle\sum_{\nu=n+1}^{2n}\frac{Y_\nu(x;\lambda)C_\nu(t)}{W} & \text{if } a \le t \le x \le b \end{cases} \quad (3.15)$$

The form of the equations (3.13)–(3.15) shows that the resolvent operator $R(\lambda; A) = (\lambda I - A)^{-1}$ is actually compact; for details of arguments of Theorem 5 in Hellwig (1967, P. 120)

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

can be used. The results of our investigations are summarized in the following expansion theorem.

THEOREM 3.1. *The linear operator A in H has an unbounded set of real eigenvalues of finite multiplicity without accumulation points in $(-\infty, \infty)$, and they can be ordered according to the size.*

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \leq \ldots \to \infty \text{ as } j \to \infty.$$

If the corresponding eigenfunctions $\psi_1, \psi_2, \ldots, \psi_j, \ldots$ form a complete orthonormal system, then for any function $f \in H$ we have the expansion

$$f = \sum_{j=1}^{\infty} < f, \psi_j >_H \psi_j, \tag{3.16}$$

in the sense of strong convergence in H.

References

Fulton, C. T. (1977) : Two-points boundary value problems with eigenvalue parameter contained in the boundary conditions, *Proc. Roy, Soc. Edin. 77A*, 293.

Hinton, D. B. (1979) : An expansion theorem for an eigenvalue problem with eigenvalue parameter in the boundary condition, *Quart. J. Math Oxford, (2)*, 33.

Hellwig, G. (1967) : *Differential Operators of Mathematical Physics*, Addison Wesely Pub. Com. U.S.A.

Schneider, A. (1974) : A note on eigenvalue problem with eigenvalue parameter in the boundary conditions, *Math. Z., 136*, 163.

Walter, J. (1973) : Regular eigenvalue problems with eigenvalue parameter in the boundary condition, *Math. Z., 133*, 301.

Zayed, E. M. E. and **Ibrahim, S. F. M.** (1989) : Eigenfunction expansion for a regular fourth order eigenvalue problem with eigenvalue parameter in the boundary conditions, *Interant, J. Math & Math. Sci., 12*, 341.

Zayed, E. M. E. (1984) : Regular eigenvalue problem with eigenvalue-parameter in the boundary conditions, *Proc. Math. Phys. Soc. Egypt., 58*, 55.

MATHEMATICS DEPARTMENT
FACULTY OF SCIENCE
ZAGAZIG UNIVERSITY
ZAGAZIG, EGYPT
AND
MATHEMATICS DEPARTMENT
FACULTY OF EDUCATION
AIN SHAMS UNIVERSITY
CAIRO, EGYPT

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc., *86*, 555–564 (1994)

SEMI-CLOSURE IN A BITOPOLOGICAL SPACE

RUPA BANDYOPADHYAY AND A. P. BAISNAB

(*Received 19 August 1993*)

Abstract. In this paper notions of semi-closure operator and pre-semi-closure operator have been introduced in a bitopological space. Their relation with Kuratowski closure operators have been explored using the concept of pairwise consistency.

1. Introduction. Let P and Q be two topologies on a non-empty set X. Then (X, P, Q) is a bitopological space. A good account of bitopological spaces can be found in J. C. Kelly's paper 'Bitopological spaces' (Kelly 1963).

N. Levine in (1963) had introduced the concepts of semi-open sets in topological space, and obtained a number of its properties while some of their bitopological analogues were studied by Shantha Bose in (1981).

S. Gena Crossley and S. K. Hildebrand had initiated the study of semi-closed sets, semi-closure operator and a pre-semi-closure operator (p.s.c. operator) in a topological space X (Crossley and Hildebrand 1971). They have also established some relations between a semi-closure operator, a p.s.c. operator and a Kuratowski closure operator on X.

Our purpose of this paper is to bring these ideas and study the same in a bitopological space. The following definitions and some of their consequences shall be useful in our study.

Let (X, P, Q) be a bitopological space.

DEFINITION 1.1. *A subset $A \subset X$ is said to be P-semi open with respect to (w.r.t)Q iff there exists a P-open set $0 \subset X$ such that $0 \subset A \subset Q - Cl(0)$. Similarly, $A \subset X$ is Q-semi open w.r.t. P iff there exists a Q-open set $0 \subset X$ such that $0 \subset A \subset P - cl(0)$.*

$A \subset X$ is said to be pairwise semi open iff it is both P-semi open w.r.t. Q and Q-semi open w.r.t. P.

As a consequence of Definitions above we have following Theorems, proofs of which are easy and left out.

THEOREM 1.1. *Following statements are equivalent in (X, P, Q).*

(i) *A is P-semi open w.r.t. Q;*

(ii) $A \subset Q - cl(P - \text{Int}(A))$;

(iii) $Q - cl(A) = Q - cl(P - \text{Int}(A))$.

THEOREM 1.2. *A union of P-semi open sets w.r.t Q is P-semi open w.r.t. Q*

REMARK 1.1. Let $P - S.0.(X)_Q$ denote the class of all P-semi open subset, of X w.r.t. Q.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Then $P \subset P - S.0.(X)_Q$

P and Q are interchangeable in all the above statements.

2. Semi-closure in (X, P, Q)

DEFINITION 2.1. *A subset A of (X, P, Q) is said to be P-semi closed w.r.t. Q iff $(X - A)$ is P-semi open w.r.t. Q.*

Similarly, Q-semi closed sets w.r.t. P and pairwise semi closed sets are defined in (X, P, Q).

We give Theorem 2.1 the proof of which is easy and left out.

THEOREM 2.1. *Following statements are equivalent in (X, P, Q).*

(1) *A is P-semi closed w.r.t. Q;*

(2) *There is a P-closed subset C of X such that Q-Int $(C) \subset A \subset C$;*

(3) *Q-Int $(P - cl(A)) \subset A$.*

DEFINITION 2.2. *For any subset A of (X, P, Q), the P-semi closure of A w.r.t. Q, denoted by ${}^P(\underline{A})_Q$ is defined by ${}^P(\underline{A})_Q = \cap_{D \in S} D$ where $S = \{D : D$ is P-semi closed w.r.t. Q and $A \subset D\}$. Similarly, P-semi interior w.r.t. Q of a subset A is defined and denoted by ${}^P(A_\circ)_Q$. Hence, a subset A of (X, P, Q) is P-semi open (semi closed) w.r.t Q iff ${}^P(A_\circ)_Q({}^P(\underline{A})_Q) = A$.*

THEOREM 2.2. *If $A \subset (X, P, Q)$, then*

(i) $X - {}^P(A_\circ)_Q = {}^P(\underline{(X - A)})_Q$;

(ii) $X - {}^P(\underline{A})_Q = {}^P((X - A)_\circ)_Q$.

Proof. (i) $X - {}^P(A_\circ)_Q$

$= X - U_{D \in \mathcal{F}} D$ where $\mathcal{F} = \{D : D$ is $P-$ semi open w.r.t. Q and $D \subset A\}$.

$= \cap_{D \in \mathcal{F}}(X - D)$

$= \cap_{E \in S} E$ where $S = \{E : E$ is $P-$semi closed w.r.t. Q and $X - A \subset X - D = E\}$.

$= {}^P(\underline{(X - A)})_Q$.

(ii) Proof is similar to that of (i).

THEOREM 2.3. *If A and B are subsets of (X, P, Q), then*

(i) ${}^P(\underline{\phi})_Q = \phi$; (ii) $A \subset {}^P(\underline{A})_Q$;

(iii) ${}^P({}^P(\underline{A})_Q)_Q = {}^P(\underline{A})_Q$; (iv) ${}^P(\underline{A \cup B})_Q \supset {}^P(\underline{A})_Q \cup {}^P(\underline{B})_Q$;

(v) ${}^P(\underline{A \cap B})_Q \subset {}^P(\underline{A})_Q \cap {}^P(\underline{B})_Q$.

Proof. The proof of (i), (ii) and (iii) are easy and left out. (iv) ${}^P(\underline{A \cup B})_Q$ is a P-semi closed set w.r.t Q containing $A \cup B$ and hence contains A. Thus, ${}^P(\underline{A})_Q \subset {}^P(\underline{A \cup B})_Q$.

Similarly, ${}^P(\underline{B})_Q \subset {}^P(\underline{A \cup B})_Q$.

Hence, ${}^P(\underline{A})_Q \cup {}^P(\underline{B})_Q \subset {}^P(\underline{A \cup B})_Q$.

(v) The proof of(v) goes similar to that of (iv).

THEOREM 2.4. *If $A \subset (X, P, Q)$, then following statements are equivalent.*

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(i) $A = {}^P(\underline{A})_Q$; (ii) ${}^Q(({}^P(\underline{A})_Q)_\circ)_P \subset A$;

(iii) $\{Q - cl(X - P - cl(A)) - (X - P - cl(A))\} \supset \{P - cl(A) - A\}$.

Proof. If $A = {}^P(\underline{A})_Q$, then ${}^Q(({}^P(\underline{A})_Q)_\circ)_P = {}^Q(A_\circ)_P \subset A$.

Hence, (i) implies (ii).

Let

$$ {}^Q\Big(({}^P(\underline{A})_Q)_\circ\Big)_P \subset A \tag{2.1}$$

Now, ${}^P(\underline{A})_Q$ is P-semi closed w.r.t. Q. So, by Theorem 2.1, there is a P-closed set C such that

$$ Q - \text{Int}\,(C) \subset {}^P(\underline{A})_Q \subset C \tag{2.2}$$

Again, Q-Int (C) is Q-semi open w.r.t. P

So,

$$ {}^Q((Q - \text{Int}\,(C))_\circ)_P = Q - \text{Int}\,(C) \tag{2.3}$$

From (2.2) and (2.3), we have

$$ Q - \text{Int}\,(C) = {}^Q((Q - \text{Int}\,(C))_\circ)_P \subset {}^Q(({}^P(\underline{A})_Q)_\circ)_P \tag{2.4}$$

From (2.1) and (2.4), we have

$$ Q - \text{Int}\,(C) \subset A $$

But $A \subset {}^P(\underline{A})_Q \subset C$.

Thus, $Q - \text{Int}\,(C) \subset A \subset C$ where C is P-closed.

By Theorem 2.1, A is P-semi closed w.r.t. Q and hence $A = {}^P(\underline{A})_Q$. Thus, (ii) implies (i).

Again, (i) and (iii) are equivalent because:

$\{Q - Cl(X - P - Cl(A)) - (X - P - Cl(A))\} \supset \{P - Cl(A) - A\}$

iff $P - Cl(A) - \{Q - Cl(X - P - Cl(A)) - (X - P - Cl(A))\} \subset A$

iff $P - Cl(A) \cap [X - \{Q - Cl(X - P - Cl(A)) - (X - P - Cl(A))\}] \subset A$

iff $P - Cl(A) \cap [X - \{Q - Cl(X - P - Cl(A)) \cap P - Cl(A)\}] \subset A$

iff $P - Cl(A) \cap [\{X - Q - Cl(X - P - Cl(A))\} \cup \{X - P - Cl(A)\}] \subset A$

iff $[P - Cl(A) \cap \{X - Q - Cl(X - P - Cl(A))\}] \cup [P - Cl(A) \cap \{X - P - Cl(A)\}] \subset A$

iff $P - Cl(A) \cap Q - \text{Int}\,(P - Cl(A)) \subset A$

iff $Q - \text{Int}\,(P - Cl(A)) \subset A$

iff A is $P-$ semi closed w.r.t. Q, by Theorem 2.1

iff $A = {}^P(\underline{A})_Q$.

THEOREM 2.5. ${}^P((P - Cl(A) - A)_\circ)_Q = \phi$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. Let us assume that ${}^P((P - Cl(A) - A)_\circ)_Q \neq \phi$.

So, there is some non-void P-semi open set w.r.t. Q, say K, such that $K \subset (P - Cl(A) - A)$.

Then there is a non-void P-open set O such that $0 \subset K \subset (P - Cl(A) - A)$.

Now $\{X - P - Cl(A)\}$ is P-open implies

$\{X - P - Cl(A)\} \cup 0$ is also P-open.

Hence $X - \{(X - P - Cl(A)) \cup 0\}$

$$= P - Cl(A) \cap (X - 0)$$
$$= P - Cl(A) - 0 \text{ is } P - \text{closed.}$$

Again, $0 \subset (P - Cl(A) - A)$ implies $A \subset (P - Cl(A) - 0)$ and $0 \neq \phi$ implies $(P - Cl(A) - 0) \subset P - Cl(A)$.

Thus, we have a P-closed set $\{P - Cl(A) - 0\}$ containing A such that $\{P - Cl(A) - 0\} \subset P - Cl(A)$ which contradicts the definition of $P - Cl(A)$.

Hence, ${}^P((P - Cl(A) - A)_\circ)_Q = \phi$.

COROLLARY. ${}^P(\underline{X - (P - Cl(A) - A)})_Q = X$

REMARK 2.1. Theorems and Definitions above remain valid when P and Q are interchanged.

3. An operator approach

DEFINITION 3.1. *For a non-empty set X, let $P(X)$ denote its power set and let $(\ )_c : P(X) \to P(X)$ be an operator satisfying*

(i) $\phi_c = \phi$; (ii) $A \subset A_c$;

(iii) $(A_c)_c = A_c$, and (iv) $(A \cup B)_c \supset A_c \cup B_c$.

Then $(\ )_c$ is called a pre-semi closure operator, abbreviated as a p.s.c. operator.

REMARK 3.1. Axiom (iv) may be replaced by $A \subset B$ implies $A_c \subset B_c$.

THEOREM 3.1. *If $(\ )_c$ and $(\cdot)_{c^*}$ are two p.s.c. operators on X, and P and Q are topologies on X for which the P-semi closed sets w.r.t. Q are $\{A : A \subset X$ and $A = A_c\}$ and the Q-semi closed sets w.r.t. P are $\{A : A \subset X$ and $A = A_{c^*}\}$; then A_c is the P-semi closure of A w.r.t. Q and A_{c^*} is the Q-semi closure of A w.r.t. P.*

Proof. Let A be a subset of X. Then $A \subset {}^P(\underline{A})_Q$ and by Remark 3.1., $A_c \subset ({}^P(\underline{A})_Q)_c$. But ${}^P(\underline{A})_Q$ is P-semi closed w.r.t. Q, so by hypothesis $({}^P(\underline{A})_Q)_c = {}^P(\underline{A})_Q$.

Hence

$$A_c \subset {}^P(\underline{A})_Q \tag{3.1}$$

As $(A_c)_c = A_c$, we have A_c as a P-semi closed set w.r.t. Q such that $A \subset A_c$.

By the minimal property of ${}^P(\underline{A})_Q$ in this respect we have

$${}^P(\underline{A})_Q \subset A_c \tag{3.2}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Thus, from(3.1) and (3.2) we have $A_c = {}^P(\underline{A})_Q$.

Hence A_c is the P-semi closure of A w.r.t. Q.

Interchanging P and Q and by similar procedure we have $A_{c^*} = {}^Q(\underline{A})_P$ and hence A_{C^*} the Q-semi closure of A w.r.t. P.

Let us consider two Kuratowski closure operators $(\ )^c$ and $(\ )^{c^*}$ on subsets of X.

Let $G = \{A \subset X : A = A^c\}$ and $G^* = \{B \subset X : B = B^{c^*}\}$

Then each of $P = \{X - A : A \in G\}$ and $Q = \{X - B : B \in G^*\}$ defines a topology on such that A^c is the P-closure of A and A^{c^*} is the Q-closure of A for every subset A of X.

NOTATION. Let A^i denotes the P-interior of A and A^{i^*} denotes the Q-interior of A.

DEFINITION 3.2. *If $(\ )^c$ and $(\ )^{c^*}$ are two Kuratowski closure operators on subsets of X, then p.s.c. operators $(\ )_c$ and $(\ )_{c^*}$ are said to be pairwise consistent with $(\ )^c$ and $(\ )^{c^*}$ if for any subset A of X,*

(i) $(A^c)^{i^*} \subset A_c, (A^{c^*})^i \subset A_{c^*}$; *and*

(ii) *If* $(A^c)^{i^*} \subset A$, *then* $A = A_c$; *and if* $(A^{c^*})^i \subset A$, *then* $A = A_{c^*}$..

THEOREM 3.2. *Let $(\ )_c$ and $(\ )_{c^*}$ be two p.s.c. operators on X that are pairwise consistent with the Kuratowski closure operators $(\ )^c$ and $(\ )^{c^*}$ on X. Then if $A \subset X$,*

(1) $(A^c)_c = A^c$;

(2) $A_c \subset A^c$;

(3) *Let* $D = \{A \subset X : A = A_c\}$ *and* $H = \{X - A : A \in D\}$.

Then

(i) $G \subset D$;

(ii) *D is the set of all P-semi closed subsets of X w.r.t. Q.*

(iii) *H is the set of all P-semi open subsets of X w.r.t. Q.*

(iv) *A_c is the P-semi closure of A w.r.t. Q.*

Proof. (1) $((A^c)^c)^{i^*} = (A^c)^{i^*} \subset A^c$.

Hence, $(A^c)_c = A^c$

(2) Now $A \subset A^c$ implies $A_c \subset (A^c)_c = A^c$

(3) (i) Let $A \in G$ so, $A = A^c$

Hence, $A_c = (A^c)_c = A^c = A$ which implies that $A \in D$.

Thus, $G \subset D$

(3) (ii) If $A \in D$, then $A = A_c$.

Using condition (i) of Definition 3.2, we have $(A^c)^{1^*} \subset A_c = A$.

This implies that Q-Int$(P - Cl(A)) \subset A$. Hence, by Theorem 2.1, A is P-semi closed w.r.t. Q.

Conversely, let B be P-semi closed w.r.t. Q. Then by Theorem 2.1, we have

$$Q - \text{Int}\,(P - Cl(B)) \subset B$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

That means $(B^c)^{i^*} \subset B$.

Hence, $B = B_c$ and $B \in D$

Thus, D is the collection of all P-semi closed subsets of X w.r.t. Q.

(3) (iii) From the construction of H it is obvious that H is the collection of all P-semi open subsets of X w.r.t. Q and $H = P - S.0(X)_Q$.

(3) (iv) This follows from Theorem 3.1. and Theorem 3.2. part (3)(ii).

COROLLARY. *In case* $(\ )_c = (\ )_{c^*}$, *then the p.s.c. operator* $(\ )_c$ *pairwise consistent with a pair of Kuratowski closure operators* $(\ )^c$ *and* $(\ )^{c^*}$ *satisfies*

(i) $(A^c)^{i^*} \subset A_c, (A^{c^*})^i \subset A_c$;

(ii) $(A^c)^{i^*} \subset A$ *implies* $A = A_c$; *and* $(A^{c^*})^i \subset A$ *implies* $A = A_c$ *for any subset* A *of* X.

Further, the collection $\{A \subset X : A = A_c\}$ gives all P-semi closed sets w.r.t. Q as well as all Q-semi closed sets w.r.t. P. And $A_c = {}^P\underline{A}_Q = {}^Q\underline{A}_P$.

EXAMPLE 3.1. Gives a p.s.c. operator, pairwise consistent with two Kuratowski closure operators on subsets of X. Take $X = \{a, b, c\}$.

For any subset A of X, we define

$$A_c = \begin{cases} X, & \text{if } a \in A \\ A, & \text{if } a \notin A \end{cases}$$

Then A_c is the P-semi closure of A w.r.t. Q as well as Q-semi closure of A w.r.t. P where $P = \{\phi\{a\}, \{a, b\}, X\}$ and $Q = \{\phi\{a\}, \{a, c\}, X\}$ are two topologies on X.

Now $(\ )_c$ is pairwise consistent with two Kuratowski closure operators $(\ )^c$ and $(\ )^{c^*}$ over X where

$$A^c = \begin{cases} X, & \text{if } a \in A \\ \{b, c\}, & \text{if } a \notin A, \quad b \in A \\ A, & \text{if } a \notin A, \quad b \notin A \end{cases}$$

$$\text{and} \quad A^{c^*} = \begin{cases} X, & \text{if } a \in A \\ \{b, c\}, & \text{if } a \notin A, \quad c \in A \\ A, & \text{if } a \notin A, \quad c \notin A. \end{cases}$$

Let us now examine the necessity of the conditions (i) and (ii) involved in Definition 3.2. of pairwise consistency of p.s..c. operators with Kuratowski closure operators. Examples 3.2. and 3.3. as cited below show the extent to which Theorem 3.2 stands in respect of necessity of conditions (i) and (ii).

EXAMPLE 3.2. Let $X = \{a, b, c\}$ with topologies P and Q where

$$P = \{\phi, \{a\}, \{a, b\}, X\} \quad \text{and} \quad Q = \{\phi, \{a\}, \{a, c\}, X\}$$

Here, the P-closed sets are $\{\phi, \{b, c\}, \{c\}, X\}$ and the Q-closed sets are $\{\phi, \{b, c\}, \{b\}, X\}$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Thus, for any subset A of X we have

$$P - Cl(Q - \text{Int}(A)) = Q - Cl(P - \text{Int}(A)) = P - \text{Int}(Q - Cl(A))$$
$$= Q - \text{Int}(P - Cl(A)) = \begin{cases} X, & \text{if } a \in A \\ \phi, & \text{if } a \notin A \end{cases}$$

$$P - S.0.(X)_Q = Q - S.0.(X)_P = \{\phi, \{a\}, \{a,b\}, \{a,c\}, X\}$$

The collection of all P-semi closed sets w.r.t. Q

$$= \text{The collection of all } Q\text{-semi closed sets w.r.t. } P$$
$$= \{\phi, \{b\}, \{c\}, \{b,c\}, X\}$$
$${}^P\underline{A}_Q = {}^Q\underline{A}_P = \begin{cases} X, & \text{if } a \in A \\ A, & \text{if } a \notin A. \end{cases}$$

We now define p.s.c. operators $(\ )_c$ and $(\ )_{c^*}$ in the manner as done by Crossley and Hildebrand in an example of (Crossley and Hildebrand 1971). That is,

$$A_c = A_{c^*} = \begin{cases} X, & \text{if } A \neq \phi \\ \phi, & \text{if } A = \phi \end{cases}$$

and let $(\ )^c$ and $(\ )^{c^*}$ be the Kuratowski closure operators corresponding to P-closure and Q-closure, respectively.

Then $Q - \text{Int}(P - Cl(A)) \subset A_c$ and $P - \text{Int}(Q - Cl(A)) \subset A_{c^*}$ for $A \subset X$.

This implies that $(A^c)^{i^*} \subset A_c$ and $(A^{c^*})^i \subset A_{c^*}$.

Thus, condition (1) of Definition 3.2. is satisfied. But condition (2) is not satisfied. Because $((\{b\}^c)^{i^*} = Q - \text{Int}(P - Cl(\{b\})) = \phi \subset \{b\}$.

But $(\{b\})_c = X \neq \{b\}$.

Also $((\{b\})^{c^*})^i = P - \text{Int}(Q - Cl(\{b\})) = \phi \subset \{b\}$.

But $(\{b\})_{c^*} = X \neq \{b\}$.

Thus, p.s.c. operators $(\ )_c$ and $(\ )_{c^*}$ are not pairwise consistent with Kuratowski closure operators $(\ )^c$ and $(\ )^{c^*}$. Here we note that ${}^P\underline{A}_Q \neq A_c, {}^Q\underline{A}_P \neq A_{c^*}$. Also here $D = \{A \subset X : A = A_c\} = \{\phi, X\}$ and this is not the collection of all P-semi closed sets w.r.t. Q.

Hence Theorem 3.2. part 3(ii) (iv) falls through.

EXAMPLE 3.3. is intended to examine the necessity of condition (i) of Definition 3.2 in relation to Theorem 3.2. This example is a modification of one as appeared in (Crossley and Hildebrand 1971) relevant to the setting of a bitopological space.

Let X be the set of all reals and let us consider the p.s.c. operators $(\ )_c$ and $(\ )_{c^*}$ each equal to the Identity operator on X. So, for any subset A of $X, A_c = A_{c^*} = A$. Let P be the usual topology on X, Q be the discrete topology on X; and $(\ )^c, (\ )^{c^*}$ be the Kuratowski closure operators corresponding to P-closure and Q-closure, respectively.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Now, if $(A^c)^{i^*} \subset A$, then $A = A_c$ and if $(A^{c^*})^i \subset A$, then $A = A_{c^*}$ for $A \subset X$.

Here condition (ii) of Definition (3.2) is satisfied but condition (i) is not satisfied Because,

If $B = ((-4,3) \cup (3,2))$, the $B^c = [-4,2]$ and $(B^c)^{i^*} = [-4,2]$

So $(B^c)^{i^*} \not\subset B$ and $B = B_c$

Hence, $(B^c)^{i^*} \not\subset B_c$.

Note that $2Cl(P - \text{Int}(A)) = P - \text{Int}(A)$. $A \in P - S.0.(X)_2$ iff $A \subset 2 - Cl(P - \text{Int}(A)) = P - \text{Int}(A)$ iff $A = P - \text{Int}(A)$.

Thus, $P - S.0(X)_Q = \{A \subset X : A \text{ is } P\text{-open}\}$. The collection of all P-semi closed sets w.r.t. $Q = \{A \subset X : A \text{ is } P\text{-closed}\}$ and ${}^P\underline{A}_Q = P - Cl(A)$.

Now for a subset A which is not P-closed in X, we have ${}^P\underline{A}_Q \neq A = A_c$.

Again the collection $D = \{A \subset X : A = A_c\}$ becomes the power set of X. So D does not give the P-semi closed sets w.r.t. Q.

Thus, Theorem 3.2 part (3) (ii) (iv) falls through.

4. Pairwise semi-closure operators in (X, P, Q) via a pair of minimal sets in X.

Given a p.s.c. operator on a non-empty set X, the notion of a minimal set in X is introduced by Crossley and Hildebrand in (1971). This section is intended to construct a pair of Kuratowski closure operators out of a given pairwise semi closure operators on X, and examine the relation of pairwise consistency amongst them.

THEOREM 4.1. (Theorem 2.10 of Crossely and Hildebrand 1971). *If $(\ )_c$ is a p.s.c. operator on X; then for every subset A of X, there exists a minimal set D_A such that*

$(A \cup D_A \cup B)_c = A \cup D_A \cup B_c$, for all subsets $B \subset X$. D_A is minimal in the sence that it is a subset of any set satisfying the above condition. For a subset A of X, put $A^k = A \cup D_A$. Crossley and Hildebrand have proved in (1971) that $(\ )^k$ is a Kuratowski closure operator on X, and further,

(i) $(A^k)_c = A^k$; (ii) $A \subset A_c \subset A^k$;

(iii) $A \cap D_A = \phi$ *and hence* $D_A = A^k - A$; (iv) $(A_c)^k = A^k$, ${}^D(A_c) = A^k - A_c$.

Before we present our theorem, we make the following definition.

DEFINITION 4.1. *Two p.s.c. operators $(\ )_c$ and $(\ )_{c^*}$ are said to be pairwise semi closure operators iff they satisfy the following conditions.*

(i) $D^*_{(X-A^k)} \supset D_A$ iff $A = A_c$; (ii) $D_{(X-A^{k^*})} \supset D^*_A$ iff $A = A_{c^*}$;

where $A^k = A \cup D_A$ and $A^{k^*} = A \cup D^*_A$. D_A is the minimal set satisfying the condition.

$(A \cup D_A \cup B)_c = A \cup D_A \cup B_c$; for $B \subset X$ and D^*_A is the minimal set satisfying the condition $(A \cup D^*_A \cup B)_{c^*} = A \cup D^*_A \cup B_{c^*}$; for $B \subset X$.

THEOREM 4.2. *If $(\ )_c$ and $(\ )_{c^*}$ are pairwise semi closure operators, $(\ )^k$ and $(\ )^{k^*}$ are the Kuratowski closure operators induced by $(\ )_c$ and $(\ )_{c^*}$ on X respectively; then $(\ )_c$ and $(\ )_{c^*}$ are pairwise consistent with $(\ )^k$ and $(\ )^{k^*}$.*

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. First we show that $(A^k)^{i^*} \subset A_c, (A^{k^*})^i \subset A_{c^*}$.
Since $(A_c)_c = A_c$ and $(A_{c^*})_{c^*} = A_{c^*}$. From Definition 4.1, we have

$$D^*_{(X-(A_c)^k)} \supset {}^D(A_c) \quad \text{and} \quad D_{(X-(A_{c^*})^{k^*})} \supset D^*_{(A_{c^*})}$$

But $(A_c)^k = A^k$ and $(A_{c^*})^{k^*} = A^{k^*}$.
Hence $D^*_{(X-A^k)} \supset D_{(A_c)} = A^k - A_c$, and $D_{(X-A^{k^*})} \supset D^*_{(A_{c^*})} = A^{k^*} - A_{c^*}$.
Now

$$\begin{aligned}(A^k)^{i^*} &= \left(X - (X - A^k)^{k^*}\right) = X - ((X - A^k) \cup D^*_{(X-A^k)}) \\ &= A^k \cap (X - D^*_{(X-A^k)}) \subset A^k \cap [X - (A^k - A_c)] \\ &= A^k \cap [X - (A^k \cap (X - A_c))] = A^k \cap [(X - A^k) \cup A_c] \\ &= (A^k \cap (X - A^k)) \cup (A^k \cap A_c) = A_c.\end{aligned}$$

Hence, $(A^k)^{i^*} \subset A_c$.
Similarly, $(A^{k^*})^i \subset A_{c^*}$.
Secondly, if $(A^k)^{i^*} \subset A$, then $A = A_c$ and if $(A^{k^*})^i \subset A$, then $A = A_{c^*}$.
As shown above $(A^k)^{i^*} = A^k \cap (X - D^*_{(X-A^k)}) = A^k - D^*_{(X-A^k)}$.
If $(A^k)^{i^*} \subset A$, then $A^k - D^*_{(X-A^k)} \subset A$.
This implies, $D^*_{(X-A^k)} \supset A^k - A = D_A$.
Hence from Definition 4.1, we have $A = A_c$. Similarly, if $(A^{k^*})^i \subset A$, then $A = A_{c^*}$.
The proof is now complete.

EXAMPLE 4.1. This is cited to support Theorem 4.2 and incidentally offers an example of a pairwise semi closure operators.

Let X be the set of all reals, also let $(\ )_c$ be the identity operator and $(\ )_{c^*}$ be defined as follows:

$$A_{c^*} = \begin{cases} X, & \text{if } A \neq \phi \\ \phi, & \text{if } A = \phi. \end{cases}$$

Clearly $(\ )_c$ and $(\ )_{c^*}$ are p.s.c. operators on X.
Now, $(A \cup D_A \cup B)_c = A \cup D_A \cup B = A \cup D_A \cup B_c$ for $B \subset X$.
Hence, $D_A = \phi$ and $A^k = A$ for $A \subset X$. The topology P induced by $(\ )^k$ is the discrete topology.
If $A = \phi$, then
$(A \cup \phi \cup B)_{c^*} = X = A \cup \phi U X = A \dot{\cup} \phi \cup B_{c^*}$, for $B \neq \phi$ and $(A \cup \phi \cup B)_{c^*} = \phi = A \cup \phi \cup B_{c^*}$, for $B = \phi$. So $D^*_A = \phi$ for $A = \phi$.
If $A \neq \phi$, then $(A \cup D^*_A \cup B)_{c^*} = X = A \cup D^*_A \cup B_{c^*}$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

If $B \neq \phi$, then $D^*_A = \phi$. If $B = \phi$, then $D^*_A = (X - A)$. So $D^*_A = (X - A)$, for $A \neq \phi$. Hence

$$A^{k^*} = \begin{cases} \phi, & \text{if } A = \phi \\ X, & \text{if } A \neq \phi \end{cases}$$
$$= A_{c^*}.$$

The topology Q induced by $(\ )^{k^*}$ is the indiscrete topology $= \{\phi, X\}$.

(i) If $A = A_c$, then $D_A = \phi \subset D^*_{(X-A^k)}$.

Conversely if, $D_A \subset D^*_{(X-A^k)}$ then from the definition of $(\ )_c$, we have $A = A_c$.

(ii) If $A = A_{c^*}$, then $A = \phi$ or X.

So, $D^*_A = \phi \subset D_{(X-A^{k^*})}$.

Conversely, if $D^*_A \subset D_{(X-A^{k^*})} = \phi$, then $D^*_A = \phi$ and hence, $A = \phi$ or X i.e., here $A = A_{c^*}$.

Hence, $(\ )_c$ and $(\ )_{c^*}$ are pairwise semi closure operators. Now, for any subset A of X, we have

$$(A^k)^{i^*} = \begin{cases} \phi, & \text{if } A \neq X \\ X, & \text{if } A = X. \end{cases}$$

So $(A^k)^{i^*} \subset A = A_c$ and $(A^{k^*})^i = A^{k^*} = A_{c^*}$.

If $(A^k)^{i^*} \subset A$, then $A = A_c$.

If $(A^{k^*})^i \subset A$, then $A_{c^*} \subset A \subset A_{c^*}$ implies $A = A_{c^*}$.

So $(\ )_c$ and $(\ )_{c^*}$ are pairwise consistent with the pair of Kuratowski closure operators as induced by them.

Acknowledgement: The first author is a C.S.I.R. research fellow and greatefully acknowledges its support in preparation of the paper.

References

Bose, Shantha (1981) : Semi open sets, semi continuity and semi open mappings in bitopological spaces, *Bull. Cal. Math. Soc.*, *73*, 237.

Crossley, S. Gena and **Hildebrand, S. K.** (1971): Semi closure, *Texas J. Sc.*, *22*. (21–3), 99.

Kelly, J. C. (1963) : Bitopological spaces, *Proc. London Math. Soc. (3)*, 13, 71.

Levine, N. (1963) : Semi open sets and semi continuity in topological spaces, *Amer. Math. Monthly*, *70*, 36.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BURDWAN
BURDWAN 713104
INDIA

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Bull. Cal. Math. Soc., *86*, 565–582 (1994)

THE NATURAL PARTIAL ORDER ON A SEMIGROUP

YU BINGJUN

(*Received 10 July 1993*)

Abstract. In this paper, the natural partial order defined by Mitsch in (1986) is further investigated. Three primitivities of elements of a semigroup and several kinds of compatibilities of the natural partial order are defined and discussed. The class of NEI-semigroups which is a proper extension of inverse semigroups is characterized and the smallest 0-restricted primitive congruence on an NEI-semigroup which is categorical at 0 is described.

1. Preliminaries, Basic Definitions and Results. In (Numbooripad 1980) the *natural partial order on a regular semigroup* is defined and investigated by Nambooripad. It is found that this order is a natural generalization of the order on inverse semigroups and plays an important role in deciding the properties and structure of a regular semigroup. In (Mitsch 1986), this order is generalized to an arbitrary semigroup and some properties of it are given. The purpose of this paper is to investigate further the properties of this generalized order and its function in deciding the properties and structures of some semigroups which are generalizations of regular ones.

This section recalls and develops some basic definitions, notations and results we will need.

Let S be an arbitrary semigroup. For any subset $T \subseteq S, E(T)$ denotes the set of all idempotents of S in T. If T is a subsemigroup of S, $RegT$ denotes the set of all regular elements of the *semigroup* T. For any $a \in RegS$, the set of all inverse elements of a is denoted by $V(a)$. The $\mathcal{R}-[resp.\mathcal{L}-, \mathcal{H}-, \mathcal{D}, \mathcal{J}-]$ class containing $a \in S$ is denoted by R_a [resp. L_a, H_a, D_a, J_a], where $\mathcal{R}, \mathcal{L}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ are the *Green's relations* on S.

The binary relation $\omega^\tau = \{(a,b) \in S \times S : \exists_s \in S^1, a = b_s\}$ and its dual, ω^l, are called the *biorder* on S. ω^τ is a quasi-order which is left compatible with the operation on S and $\omega^\tau \cap (\omega^\tau)^{-1} = \mathcal{R}$. The dual results hold for ω^l. Let $\omega = \omega^\tau \cap \omega^l$. Then the following result and its dual hold.

RESULT 1. For $e \in E(S)$ and $a \in S$,

$$(a,e) \in \omega^\tau \Longleftrightarrow a = ea, \quad \text{and} \quad (e,a) \in \omega^\tau \Rightarrow ae \in R_e \cap \omega(a).$$

where, $\omega(a) = \{x \in S : (x,a) \in \omega\}$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The *natural partial order*, NPO on S, denoted by $\leq$ or $\leq_s$ is defined as follows:

$a \leq b \Longleftrightarrow a\omega^{\tau}b$ and, for some $x \in S^1, a = xb = xa, \quad \forall a, b \in S$.

The following two results can be found in (Mitsch 1986)

RESULT 2. *$\leq$ is a partial order on S. And for any $a, b \in S$ the following three statements are equivalent:*

(1) $a \leq b$;

(2) $a\omega^1 b$ and, for same $y \in S^1, a = by = ay$;

(3) $a = xb = xa = by = ay$, for some $x, y \in S^1$.

Further, if $b \in RegS$, the foregoing three statements are equivalent to the following:

(4) $a\omega b$ and for some (all) $b' \in V(b), a = ab'a$;

(5) For any $f \in E(R_b), \exists e \in E(R_a)$ such that $e\omega f$ and $a = eb$;

(6) For any $f' \in E(L_b), \exists e' \in E(L_a)$ such that $e'\omega f'$ and $a = be'$;

(7) $a = be = fb$ for some $e, f \in E(S)$.

RESULT 3. *We have*

(1) $\leq \cap \mathcal{R} = \leq \cap \mathcal{L} = 1s$;

(2) $E(S)$ and $RegS$ are both order ideals with respect to $\leq$.

(3) The restriction of $\leq$ to $E(S)$ coincides with ω which is also the usual partial order on $E(S)$.

Define binary relations on a semigroup S as follows:

$$\mathcal{R}^* = \{(a, b) \in S \times S : xa = ya \Longleftrightarrow xb = yb \quad \forall x, y \in S^1\};$$
$$\tilde{\mathcal{R}} = \{a, b) \in S \times S : xa\mathcal{L}ya \Longleftrightarrow xb\mathcal{L}yb, \quad \forall x, y \in S^1\}.$$

The following lemma is a result in (Fountain 1982) and (Xiangdong 1989).

LEMMA 4. *We have*

(1) *$\mathcal{R} \subseteq \mathcal{R}^* \subseteq \tilde{\mathcal{R}}$ and these inclusions are generally strict, but*

$$\mathcal{R}^* \cap (RegS \times RegS) = \mathcal{R} \cap (RegS \times RegS).$$

(2) *$\mathcal{R}^*$ and $\tilde{\mathcal{R}}$ are both left congruences on S.*

Dually, we can define $\mathcal{L}^*$ and $\tilde{\mathcal{L}}$ for which the dual properties hold.

PROPOSITION 5. *Let Sbe a semigroup, we have $\leq \cap \tilde{\mathcal{R}} = \leq \cap \tilde{\mathcal{L}} = 1s$.*

Proof. Let $(a, b) \in \leq \cap \tilde{\mathcal{R}}$, there is $x \in S^1$ such that $a = xb = xa$. Because of $a\tilde{\mathcal{R}}b$ we have $b\mathcal{L}xb = a$. From the Result 3 we have $a = b$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

According to (Yu Bingjun 1993), the following three subsets $E(S)^{n-1}, n > 1$

$$\begin{aligned}
\mathcal{M}(a_1,\ldots,a_n) = \{(g_1,\ldots,g_{n-1}) &: \\
g_i \in E(S)) \cap \omega^1(a_i) \cap \omega^\tau(a_{i+1}), i = 1,\ldots,n-1, \\
g_j a_{j+1} = a_{j+1} g_{j+1}, j = 1,\ldots,n-2\} \\
\mathcal{S}(a_1,\ldots,a_n) = \{(h_1,\ldots,h_{n-1}) \in \mathcal{M}(a_1,\ldots,a_n) &: \\
\forall (a_1,\ldots,g_{n-1}) \in \mathcal{M}(a_1,\ldots,a_n), \\
a_1 g_1 \omega^\tau a_1 h_1, g_{n-1} a_n \omega^1 h_{n-1} a_n\}, \\
\mathcal{S}_1(a_1,\ldots,a_n) = \{(h_1,\ldots,h_{n-1}) \in \mathcal{M}(a_1,\ldots,a_n) &: \\
a_1 h_1 a_2 h_2 \ldots a_{n-1} h_{n-1} a_n = a_1 a_2 \ldots a_{n-1} a_n\}
\end{aligned}$$

are called the M-set, sandwich set and inner sandwich set of $(a_1,\ldots,a_n)$ in S, respectively. And the next result holds:

RESULT 6. *Let $(a_1,\ldots,a_n)$ be sequence of n elements of a semigroup S. For any $(h_1,\ldots,h_{n-1}) \in \mathcal{M}(a_1,\ldots,a_n)$, we have*

(1) $a_1 \ldots a_i h_i a_{i+1} \ldots a_n = a_1 h_1 a_2 h_2 \ldots a_{n-1} h_{n-1} a_n, \forall i = 1,\ldots,n-1;$

(2) $a_1 h_1 \mathcal{R} a_1 h_1 a_2 h_2 \ldots h_{n-1} a_n \mathcal{L} h_{n-1} a_n;$

Further we have

(3) $S_1(a_1,\ldots,a_n) \subseteq S(a_1,\ldots,a_n)$ *and if* $S_1(a_1,\ldots,a_n) \neq \phi$ *then* $S_1(a_1,\ldots,a_n) = S(a_1,\ldots,a_n)$.

(4) $S_1(a_1,\ldots,a_n) \neq \phi \Longleftrightarrow a_1 \ldots a_n \in RegS;$

(5) *If* $S(a_1,\ldots,a_n) \neq \phi$, *then* $S^{(i)}(a_1,\ldots,a_n)$ *is a rectangular sub-band of* S, *where* $S^{(i)}(a_1,\ldots,a_n)$ *is the set of the i-th components of all the elements in* $S(a_1,\ldots,a_n), i = 1,\ldots,n-1$.

The following proposition shows that there is a close relationship between the natural partial order and the foregoing three sets:

PROPOSITION 7. *Let S be a semigroup with $regS \neq \phi$. For any $a_1,\ldots,a_n \in S, n \geq 2$, if $\mathcal{M}(a_1,\ldots,a_n) \neq \phi$, then there exist $b_1,\ldots,b_n \in RegS$ such that $b_i \leq a_i, i = 1,\ldots,n$. If further $a_1 \ldots a_n \in RegS$ then for each $i = 1,\ldots,n, b_i \mathcal{D} a_1 \ldots a_n$, where $\mathcal{D}$ is the Green's relation $\mathcal{D}$ on S.*

Proof. Take $(h_1,\ldots,h_{n-1}) \in \mathcal{M}(a_1,\ldots,a_n)$. Let $b_j = a_j h_j, j = 1,\ldots,n-1$ and $b_n = h_{n-1} a_n$. By Result 1 we have $b_i \in RegS, i = 1,\ldots,n$ and for any $j = 2,\ldots,n-1, b_j = a_j h_j = h_{j-1} a_j$. Since $h_j \in E(S), i = 1,\ldots,n-1$, we have $b_j \leq a_j$ by Result 2. Moreover, we have $b_1 = a_1 h_1 \mathcal{L} h_1 \omega^1 a_1$ and $b_n = h_{n-1} a_n \mathcal{R} a_n$, thus $b_1 \leq a_1$ and $b_n \leq a_n$.

If $a_1 \ldots a_n \in RegS$, we can take $(h_1,\ldots,h_{n-1}) \in S_1(a_1,\ldots,a_n)$. In this case we have $b_1 \mathcal{L} h_1 \mathcal{R} b_2 \mathcal{L} \ldots \mathcal{L} h_{n-1} \mathcal{R} b_n \mathcal{L} a_1 \ldots a_n$. Hence $b_i \mathcal{D} a_1 \ldots a_n$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

2. Some Further Elementary Properties of NPO. An equivalence relation ρ on a partially ordered set $(S, \leq)$ is said to be reflective with respect to $\leq$, if for any a, b and $c \in S$ with $a \leq b\rho c$ there is $b' \in S$ such that $a\rho b' \leq c$.

PROPOSITION 8. *The Green's relations $\mathcal{R}, \mathcal{L}$ and $\mathcal{D}$ on a semigroup S are all reflective with respect to the natural partial order $\leq$ on S. More precisely, denoting any one of them by $\mathcal{K}$, then for any a, b and $c \in S$ with $a < b\mathcal{K}c$, there exists $b' \in S$ such that $a\mathcal{K}b' < c$, here $<$ means "$\leq$ but $\neq$".*

Proof. Let $a < b\mathcal{R}c$. We have $a\omega^\tau b$ and there are $x, u, v \in S^1$ such that $a = xb = xa, b = cu$ and $c = bv$. Let $b' = av$, we have $b'u = avu = xbuv = xcu = xb = a$ which implies $a\mathcal{R}b'$. Furthermore, we have $b'\omega^\tau a\omega^\tau b\mathcal{R}c$, and $b' = av = xav = xb' = xbv = xc$. Hence $b' \leq c$. If $b' = c$, we have $a = xb = xbvu = avu = b'u = cu = b$. This proves that $a \neq b$ implies $a\mathcal{R}b' < c$.

The statement $a < b\mathcal{L}c \Rightarrow \exists b' \in S, a\mathcal{L}b' < c$ can be proved dually.

Now, let $a < b\mathcal{D}c$. There exists $d \in S$ such that $a < b\mathcal{R}d\mathcal{L}c$. From the above results, there are $b', d' \in S$ such that $a\mathcal{R}d' < d\mathcal{L}c$ and $a\mathcal{R}d'\mathcal{L}b' < c$, that is, $a\mathcal{D}b' < c$.

The *trace product* $*$ on regular semigroups defined in (Nambooripad 1980) can be easily defined on the set $RegS$ of any semigroup S with $RegS \neq \phi$ as follows: for any $x, y \in RegS$

$$x * y = \begin{cases} xy, & \text{if } xy \in R_x \cap L_y, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

The following lemma is a consequence of Miller-Clifford Theorem (1961 and 1967).

LEMMA 9. *For any $x_1, \ldots x_n \in RegS, n \geq 2$, define inductively*

$$x_1 * x_2 * \ldots * x_n = \begin{cases} (x_1 * \ldots * x_{n-1}) * x_n, & \text{both } (x_1 * \ldots * x_{n-1}) \\ & \text{and } (x_1 * \ldots * x_{n-1}) * x_n \text{ exist,} \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Then we have

(1) *$x_1 * x_2 * \ldots * x_n$ is well-defined if and only if there exist $h_i \in E(S)$ such that $x_i\mathcal{L}h_i\mathcal{R}x_{i+1}, i = 1, \ldots, n-1$. If this is the case, $x_1 * x_2 * \ldots * x_n = x_1x_2 \ldots x_n$;*

(2) *If $x_1 * x_2 * \ldots * x_n$ is well-defined, giving parentheses in any way to this product, then every $*$ product obtained is also well-defined and the final product is the same.*

(3) *If $x_1 * x_2 * \ldots x_n$ is well-defined, then $x_1\mathcal{R}x_1 * x_2 * \ldots * x_n\mathcal{L}x_n$.*

PROPOSITION 10. *Let S be a semigroup with $RegS \neq \phi$. For any $x_1, \ldots, x_n \in S, n \geq 2$, if $x_1 \ldots x_n \in RegS$ then there exist $y_1, \ldots, y_n \in RegS$ satisfying the following conditions:*

(1) $y_i \leq x_{ij}$; (2) $x_1x_2 \ldots x_n = y_1 * y_2 * \ldots * y_n$.

Further, if $x_i \in RegS\ \forall_i = 1, \ldots, n$ and $x_1 * x_2 * \ldots x_n$ is well-defined, then for any $y_1, \ldots, y_n \in RegS$ satisfying the two conditions above we have $y_i = x_i, \forall i = 1, \ldots, n$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. Since $x_a \ldots x_n \in RegS$, there is $(h_1, \ldots, h_{n-1}) \in S_1(a_1, \ldots, a_n)$ from Result 6. Let $y_i = x_j h_j, j = 1, \ldots n-1$, and $y_n = h_{n-1}x_n$, one verifies that the conditions (1) and (2) are satisfied.

If $x_i \in RegS, i = 1, \ldots, n$ and $x_1 * x_2 * \ldots * x_n$ is well-defined, from Lemma 9 we have $x_1 \mathcal{R} x_1 * x_2 * \ldots * x_n = x_1 \ldots x_n \mathcal{L} x_n$. If $y_1, \ldots, y_n \in RegS$ satisfy the conditions (1) and (2), we prove inductively $y_k = x_k$ as follows:

First, from $y_1 \leq x_1$ and $x_1 \mathcal{R} x_1 * x_2 * \ldots * x_n = x_1 \ldots x_n = y_1 * y_2 * \ldots * y_n \mathcal{R} y_1$ we have $(y_1, x_1) \in \leq \cap \mathcal{R} = 1s$, i.e., $x_1 = y_1$.

Now let $y_{k-1} = x_{k-1} k \geq 2$. From Lemma 9 there are $e_{k-1}, f_{k-1} \in E(S)$ such that $x_{k-1} \mathcal{L} e_{k-1} \mathcal{R} x_k$, and $y_{k-1} \mathcal{L} f_{k-1} \mathcal{R} y_k$. So $e_{k-1} \mathcal{L} f_{k-1}$. Since $y_k \leq x_k$ and $e_{k-1} \in E(R_{x_k})$, there is $g \in E(R_{y_k})$ such that $g \omega e_{k-1}$ and $y_k = g x_k$ by Result 2(5). Thus, we have $g \mathcal{R} f_{k-1}$ and $e_{k-1} = e_{k-1} f_{k-1} = e_{k-1} g f_{k-1} = g f_{k-1} = f_{k-1}$. This proves $(y_k, x_k) \in \leq \cap \mathcal{R} = 1s$, i.e., $y_k = x_k$, completing the proof.

3. Three Primitivities of Elements of a Semigroup. It is well-known that the primitivity of idempotents plays an important role in characterizing the structure of regular semigroups. In (Nambooripad 1980), this primitivity is extended to any regular elements via the NPO and some new properties of regular semigroups are obtained. Here we extend further the primitivity to elements of an arbitrary semigroup and apply them to investigate the properties of the semigroup.

DEFINITION 1. *Let S be a semigroup. A non-zero element $a \in S$ is said to be $[\mathcal{J}-, \mathcal{D}-]$ primitive, if a is a minimal element in the set $[J_a, D_a], S - \{0\}$ with respect to the natural partial order on S.*

From the Proposition 8 we immediately have

PROPOSITION 11. *If $a \in S$ is [res. $\mathcal{J}-, \mathcal{D}-$] primitive, then every element in D_a is [res. $\mathcal{J}-, \mathcal{D}-$] primitive.*

A semigroup S is called primitive, if every non-zero element in S is primitive. If this is the case and $RegS$ has at most one element, S is called primitive regular-free.

If a semigroup S is an ideal extension of a semigroup N by a semigroup R° with zero, S is called a primitive extension of N by R°, if the following axiom (PE) holds:

(PE) For any non-zero elements $a, b \in N$, any $x \in R$ and $y \in S$, either of the following two equations implies $a = b$:

(1) $a = bx = yb = ya$;

(2) $a = xb = by = ay$.

PROPOSITION 12. A semigroup S is primitive if and only if S is one of the following three kinds of semigroups:

(1) A primitive regular semigroups;

(2) A primitive regular-free semigroup;

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(3) A primitive extension of a primitive regular-free semigroup by a primitive regular one.

Proof. Assume S has zero. (When S has no zero, the proof is similar.) Let S be primitive. If S is regular or, conversely, it has no non-zero regular element, the result naturally holds. Now, let S not be regular but $RegS$ contains non-zero element.

Let $N = S - RegS$ and $N^\circ = N \cup \{0\}$ where 0 is the zero element of S. For any $a \in N$ and $b \in S$, if $ab \in S - N^\circ = RegS - \{0\} = R_1$ by Proposition 7 there is $a' \in RegS$ such that $a' \leq a$ and $a'\mathcal{D}ab$. Since S is primitive, we have $a' = 0$ so that $ab = 0$ which is impossible. This shows that N° is a right ideal of S. Clearly, N° is a left ideal of S. Similarly, N° is a primitive regular-free semigroup.

Up to isomorphism, the Rees quotient S/N° can be viewed as $R^\circ = R \cup \{0\}$, where 0 is a symbol not in S, on which the multiplication is defined as follows: $\forall a, b \in R$.

$$a \cdot b = \begin{cases} ab, & \text{if } ab \in RegS; \\ 0, & \text{otherwise.} \end{cases}$$
$$a \cdot 0 = 0 \cdot a = 0 \cdot 0 = 0.$$

Evidently R°, is a regular semigroup and from Result 2 we have

$$\begin{aligned} \leq_{R^\circ} &= (\leq s \cap (R \times R)) \cup (0, R^\circ) \\ &= ((1_{s-\{0\}} \cup (0, S)) \cap (R \times R)) \cup (0, R^\circ) \\ &= 1_R \cup (0, R^\circ) \end{aligned}$$

This shows that $R^\circ \cong S/N^\circ$ is a primitive regular semigroup. From the primitivity of S, the axiom (PE) clearly holds. Thus, S is a primitive extension of the primitive regular-free semigroup N° by the primitive regular semigroup R°.

Conversely, it suffices to prove that if S is of the form (3), then S is a primitive semigroup. So let S be a primitive extension of a primitive regular free semigroup $N^\circ = N \cup \{0\}$ by a primitive regular semigroup $R^\circ = R \cup \{0\}$. For any $a, b \in S - \{0\}$ with $a \leq_s b$, there are exactly two cases:

CASE 1. $b \in R$. In this case $a \in \subseteq R$ and $a = be = bf$ for some $e, f \in E(S) - \{0\}$ by Result 2. Thus, $a \leq_{R^\circ} b$, for $a, b, e, f \in R^\circ$. From the primitivity of R°, we have $a = b$.

CASE 2. $b \in N$. Since N° is an ideal of S, we have $a \in N$ and for some $x, y \in S^1, a = bx = ax = yb = ya$. If both of $x, y \in N$, we have $a \leq_{N^\circ} b$ and $a = b$ by the primitivity of N°. Otherwise at least one of the equations in (PE) holds and $a = b$ from the axiom (PE) completing the proof.

The following example shows that not every ideal extension of a primitive regular-free semigroup by a primitive regular one is a primitive semigroup.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

EXAMPLE 1. Let both of A and B be infinite cyclic semigroups generated by a and b respectively, C a group of order 2 generated by c which are pairwise disjoint. Let $S = A \cup B \cup C$, define a multiplication on S extending those on A, B and C as follows: $c^k a^m = a^m c^k = c^k b^m = b^m c^k = a^m, a^m b^n = b^n a^m = a^{m+n}$, where, $k = 1, 2, m \geq 1, n \geq 1$. One verifies that S is a semigroup in which $N = A \cup B$ is a primitive regular-free ideal of S and $R^e = S/N$ a group with zero adjoined. Thus, S is an ideal extension of a primitive regular-free semigroup by a primitive regular semigroup. However, S is not a primitive semigroup, since we have $0 \neq a < b$.

Besides, $J_b = \{b\}$ in S, so b is $\mathcal{J}-$ primitive but not primitive. This shows that the primitivity and the $\mathcal{J}$-primitivity are different.

According to (Fountain 1982), a semigroup S is called left [right] abundant, if every $\mathcal{L}^* - [\mathcal{R}^* -]$ class of S contains an idempotent. The class of all left [right] abundant semigroups is a proper extension of regular ones but it possesses many properties possessed by the latter. The following is one.

PROPOSITION 13. *A left [right] abundant semigroup S is primitive if and only if every non-zero idempotent of S is primitive.*

To prove this proposition we first give two lemmas.

LEMMA 14. *Let S be a left abundant semigroup. For any $a, b \in S, a \leq b$ if and only if $a\omega^l b$ and $a = be$, for some $e \in E(L_a^*)$, where L_a^* is the $\mathcal{L}^*$-class containing a.*

Proof. It suffices to prove the necessity. Let $a \leq b$. Then $a\omega^l b$ and $a = bx = ax$ for some $x \in S^1$. Since S is left abundant, there is $g \in E(L_a^*)$ such that $a = ag$. From $a\mathcal{L}^* g$ we have $g = gx$. Write $e = xg$. We have $e^2 = xgxg = xg^2 = xg = e, eg = e$ and $ge = gxg = g^2 = g$. Thus, $e \in E(L_g) = E(L_g^*) = E(L_a^*)$ and $a = ag = bxg = be$.

LEMMA 15. Let S be a left abundant semigroup. For any $a, b, c \in S$, if $a \leq b$ and $c \in L_b^* \cap RegS$ then there is $b' \in L_a^*$ such that $b' \leq c$; Further, if $a \neq b$ then $b' \neq c$.

Proof. From Lemma 14 there is $e \in E(L_a^*)$ and $y \in S^1$ such that $a = be = yb$. Take $g \in E(L_b^*)$. We have $bg = b$, so $ag = ybg = yb = a$. But $a\mathcal{L}^* e$ we have $eg = e$. Let $h = ge$ then $h \in E(L_e) = E(L_a^*)$ and $h\omega g$. Because $(c, g) \in \mathcal{L}^* \cap (RegS \times RegS) \subseteq \mathcal{L}$, we have $h \leq g\mathcal{L}c$. From Proposition 8 there is $b' \in S$ such that $h\mathcal{L}b' \leq c$, clearly $b' \in L_h^* = L_a^*$.

If $a \neq b$ by Lemma 4 and Proposition 5 we have $L_a^* \neq L_b^*$ so that $h < g\mathcal{L}c$. From Proposition 8 again we have $h\mathcal{L}b' < c$.

Proof of Proposition 13. It suffices to prove the sufficiency for left abundant semigroup. So let S be left abundant in which every nonzero idempotent is primitive. If $a < b$ for some $a, b \in S$, take $e \in E(L_b^*)$, by Lemma 15 there is $f \in L_a^*$ with $f < e$. Because $E(S)$ is an order ideal of S and e is primitive, we have $f = 0$ so that $a = 0$, completing the proof.

The next proposition shows that, like groups primitive semigroups are also basic structural units in every semigroup with idempotents.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

PROPOSITION 16. *Let S be a semigroup with $E(S) \neq \phi$. Denote $\tilde{\mathcal{H}} = \tilde{\mathcal{R}} \cap \tilde{\mathcal{L}}$. For any $e \in E(S)$, let $\tilde{\mathcal{H}}_e = \{x \in S : x\tilde{\mathcal{H}}e\}$. We have*

(1) *$\tilde{\mathcal{H}}_e$ is a subsemigroup of S*

(2) *For any $a \in \tilde{H}_e, a$ is a regular element of the semigroup $\tilde{H}_e$ if and only if the $\mathcal{H}$-class H_a of the semigroup S is a group.*

(3) *$\tilde{H}_e$ is a completely simple semigroup or primitive extension of a primitive regular-free semigroup without zero by a completely simple semigroup with a zero adjoined.*

Proof. (1) Let $a, b \in \tilde{H}_e$. We have to prove $ab \in \tilde{H}_e$. For any $u, v \in S^1$ with $abu\mathcal{R}abv$, we have $ebu\mathcal{R}ebv$ from $a\tilde{\mathcal{L}}e$. Since $b\tilde{\mathcal{R}}e$ and $ee\mathcal{L}1.e$ we have $eb\mathcal{L}b$. So there is $b_1 \in S^1$ such that $b = b_1eb$. Because $\mathcal{R}$ is a left congruence and $ebu\mathcal{R}ebv$, we obtained $bu = b_1ebu\mathcal{R}b_1ebv = bv$. Thus, $eu\mathcal{R}ev$ by $b\tilde{\mathcal{L}}e$. Conversely, if $eu\mathcal{R}ev$, we have $bu\mathcal{R}bv$ by $b\tilde{\mathcal{L}}e$. Thus, $abu\mathcal{R}abv$ since $\mathcal{R}$ is a left congruence on S. This proves $ab\tilde{\mathcal{L}}e$. Dually we can prove $ab\tilde{\mathcal{R}}e$.

(2) From Lemma 4, $\mathcal{H} \subseteq \tilde{\mathcal{H}}$ so that $H_a \subseteq \tilde{\mathcal{H}}_e$ for any $a \in \tilde{H}_e$, where H_a denotes the $\mathcal{H}$-class of S containing a. Thus, if H_a is a group, a is a regular element of the semigroup $\tilde{H}_e$. Conversely, assume $a \in Reg\tilde{H}_e$, there are $f, g \in E(\tilde{H}_e)$ such that $f\mathcal{L}'a\mathcal{R}'g$, where $\mathcal{L}'$ and $\mathcal{R}'$ are the Green's relations on $\tilde{H}_e$. Of course, $e\mathcal{L}a\mathcal{R}g$ in S. From $(f, g) \in \tilde{\mathcal{H}} \cap (E(S) \times E(S))$ one verifies $fg \in R_f \cap L_g$. Thus, there is $h \in E(S)$ such that $h \in L_f \cap R_g = L_a \cap R_a = H_a$ which means H_a is a group.

(3) From (2) we see that $Reg\tilde{H}_e = \cup\{H_f : f\tilde{\mathcal{H}}_e, f^2 = f, f \in S\}$. By the definitions of $\tilde{\mathcal{L}}, \tilde{\mathcal{R}}$ and the Miller-Clifford Theorem, one easily verifies that $Reg\tilde{H}_e$ is a completely simple semigroup. As a consequence of Proposition 5, $\tilde{H}_e$ is either a singleton or a primitive semigroup without zero. If $N_e = \tilde{H}_e - Reg\tilde{H}_e$ is empty, then $\tilde{H}_e$ is a completely simple semigroup. Otherwise, N_e is a primitive regular-free semigroup without zero and $\tilde{H}_e$ is a primitive extension of N_e by $Reg\tilde{H}_e^0$ which is a completely simple semigroup with a zero adjoined. This completes the proof.

PROPOSITION 17. *Let S be a semigroup, $a \in RegS$ and $b \in S - RegS$. Then*

(1) *a is $\mathcal{J}$-primitive if and only if a is $\mathcal{D}$-primitive;*

(2) *if b is J-primitive, then $J_b \cap RegS = \phi$.*

Proof. (1) It suffices to prove that the $\mathcal{D}$-primitivity of a implies the $\mathcal{J}$-primitivity of it. So assume that a is $\mathcal{D}$-primitive and $x \in J_a$ with $x \leq a$. Since $x\mathcal{J}a$ there are $u, v \in S^1$ such that $a = uxv$. If $u = v = 1$ we are done. Otherwise one of the M-sets $\mathcal{M}(u, x), \mathcal{M}(u, x, v)$ and $\mathcal{M}(x, v)$ is not empty from Result 6. By Proposition 7$y \leq x \leq a$ for some $y\mathcal{D}a$ which implies $y = x = a$ because of the $\mathcal{D}$-primitivity of a.

(2) If $a \in J_b \cap RegS$, then $a = ubv$ for some $u, v \in S^1$. Using the same argument as above shows that $a' \leq b$ for some $a'\mathcal{D}a$. Since $b \in S - RegS, a' \neq b$. However, $a'\mathcal{J}a\mathcal{J}b$ contradicts the $\mathcal{J}$-primitivity of b.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The next proposition is in a sense a generalization of an Anderson's theorem ((Clifford and Preston 1967 and1967 Theorem 2.54).

PROPOSITION 18. *Let J be a $\mathcal{J}$-class of a semigroup S. If J contains both regular and nonregular elements of S, then no element in J is $\mathcal{J}$-primitive. In particular, every idempotent in J is the identity of a copy of the bicyclic semigroup.*

Proof. It suffices to prove that for any idempotent e in J, e is not $\mathcal{J}$-primitive. In fact, if e is $\mathcal{J}$-primitive, e is a primitive idempotent in the principal factor $J(e)/I(e)$, where $J(e)$ is the principal ideal of S generated by e and $I(e) = \{x \in J(e) : J(x) \neq J(e)\}$. Thus, $J(e)/I(e)$ is completely simple or completely 0-simple which implies that $J(e)/I(e)$ is bisimple or 0-bisimple. From this we have $D_e = J_e = J$ which implies that J contains only regular elements of S.

Further, e is not $\mathcal{D}$-primitive by Proposition 17. So there is $f \in D_e$ such that $f < e$. Taking $p \in R_e \cap L_f$ and $q \in L_e \cap R_f \cap V(p)$, one verifies that the subsemigroup of S generated by p, q is isomorphic to the bicyclic semigroup and that e is the identity of this subsemigroup.

At the end of this section, we give an example to show that the $\mathcal{J}-$ and $\mathcal{D}$-primitivities are generally different from each other.

EXAMPLE 2. Let $X = \{a, b, c\}$ and X^* the free moniod over X in which the identity is denoted by 1. Let ρ be the congruence on X^* generated by the binary relation $\{(abc, 1)\}$ and $S = X^*/\rho$. Then we have the following:

PROPOSITION 19. *In the semigroup S in Example 2 above, the element $b\rho$ is a $\mathcal{D}$-primitive element which is not $\mathcal{J}$-primitive.*

Proof. It suffices to prove the following three statements among which the first two are easily verified.

(1) For any $w \in X^*$ with length greater than 1, if $w\rho b$, then the first and the final letters in w must be different from each other. Thus, $b\rho$ is a nonregular element of S;

(2) We have $b\rho\mathcal{J}1\rho, (bcah)\rho\mathcal{D}1\rho$ in S and $(bcab)\rho \leq b\rho$ in S. Therefore, $b\rho$ is not $\mathcal{J}$-primitive;

(3) For any $w \in X^*$, if $w\rho < b\rho$ in S, then $w\rho \in RegS$. Therefore, $b\rho$ is $\mathcal{D}$-primitive.

Now let us prove the statement (3). Let $w\rho < b\rho$. There exist $x, y \in X^+$ such that $w\rho = b\rho x\rho = y\rho b\rho = w\rho x\rho$. Thus, $(bx, yb) \in \rho$ which means that there exist $w_o, w_1, \ldots, w_n \in X^+, n \geq 1$ and $s_i, t_i \in X^*, i = 0, 1, \ldots, n$, such that

$$w_o = bx, \qquad w_n = yb,$$

$$w_i = s_i t_i, \qquad w_{i-1} = s_i abct_i \quad \text{or}$$

$$w_i = s_{i-1} abct_{i-1}, \qquad w_{i-1} = s_{i-1} t_{i-1}.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

From this we can prove in induction on $i = 0, 1, \ldots, n$ that there exist $v_i, u_i \in X^*$ with $u_i \rho 1$ such that $w_i = v_i b u_i$ in X^*.

The result clearly holds for $i = n$. Assume $w_i = v_i b u_i, v_i, u_i \in X^*$ with $u_i \rho 1, 1 \leq i \leq n$. From the expression of w_{i-1} and w_i above, noting that $u_i = 1$ or has a its first letter by $u_i \rho 1$, we have

$$\begin{aligned} w_{i-1} &= v_{i-1} b u_{i-1}, && \text{where,} \\ u_{i-1} &= u'_{i-1} u''_{i-1} && \text{and } u_i = u'_{i-1} abcu''_{i-1}, \text{ or} \\ u_{i-1} &= u'_i abcu''_i && \text{and } u_i = u'_i u''_i, \text{ or} \\ u_{i-1} &= u_i, \end{aligned}$$

where u'_i, u'_{i-1}, u''_i and $u''_{i-1} \in X^*$. In any case we have $u_{i-1} \rho u_i \rho 1$.

In particular, we have $w_\circ = bx = v_\circ b u_\circ$ in X^* with $v_\circ, u_\circ \in X^*$ and $u_\circ \rho 1$. Since $w_\circ \rho = w\rho \neq b\rho$ and $u_\circ \rho = 1\rho$, we have $v_\circ \neq 1$, whence $v_\circ = bv'$ and $x = v' b u_\circ$ for some $v' \in X^*$. Therefore, we have

$$\begin{aligned} w\rho v' \rho w \rho &= (ybv'bx)\rho = (ybv'bu_\circ x)\rho = (ybxx)\rho \\ &= w\rho x \rho x \rho = w\rho \end{aligned}$$

in S. This completes the proof.

4. Several Compatibilities of NPO w.r.t. the Operation. As in (Blyth and Gomes 1983), (Nambooripad 1980) the NPO $\leq$ on a semigroup S is called right [left] compatible w.r.t. the operation on S, if for any $a, b, c \in S, a \leq b$ implies $ac \leq bc[ca \leq cb]$. It is called compatible, if it is both right and left compatible. Some sufficient and necessary conditions for various compatibilities on regular semigroups have been given in (Blyth and Gomes 1983) and (Nambooripad 1980). Here, we discuss some local compatibilities of NPO related to the regular elements of an arbitrary semigroup. The main results on regular semigroups are generalized and some important differences are discovered.

Let S be a semigroup with $RegS \neq \phi, \leq$ the NPO on it.

(1) $\leq$ is called [super-] regularly right compatible w.r.t. the operation on S, shortened as $[S]RR$-compatible, if for any $a, b \in RegS$ and $c \in RegS[S], a \leq b$ implies $ac \leq bc$;

(2) $\leq$ is called weakly super-regularly right compatible w.r.t. the operation on S, shortened as WSRR-compatible, if for any $a, b \in RegS$ and $c \in S, a \leq b$ and $ac \in RegS$ together imply $ac \leq bc$.

The $[S]LR$-and WSLR-compatibilities can be defined dually. Further, $\leq$ is called $[S]R$-compatible, if it is both $[S]RR$ and SLR compatible. It is WSR compatible, if it is both WSRR and WSLR-compatible.

For brevity, we define the following notions:

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Let S be a semigroup with $E(S) \neq \phi$.

(1) S is called $\mathcal{L}-[\mathcal{R}-]$ unipotent, if there is exactly one idempotent in every regular $\mathcal{L}-[\mathcal{R}-]$ class of S; S is called locally $\mathcal{L}-[R-]$ unipotent, if for any $e \in E(S)$, subsemigroup eSe is $\mathcal{L}-[\mathcal{R}-]$ unipotent.

(2) S is called subinverse, if $E(S)$, is a semilattice; S is called locally subinverse, if for any $e \in E(S), eSe$ is subinverse. In particular, a locally subinverse regular semigroup is simply called locally inverse.

PROPOSITION 20. *The following statements concerning a semigroup S with $RegS \neq \phi$ satisfy the implications*

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4).$$

(1) $\leq$ is RR-compatible.

(2) For any $a \in S$ and $b \in RegS$, whenever $S_1(a,b) \neq \phi$, it is a right zero semigroup.

(3) S is locally $\mathcal{L}$-unipotent.

(4) $\leq$ is WSRR-compatible.

Proof. (1) $\Rightarrow$ (2), Let $S_1(a,b) \neq \phi$. For any $f \in E(R_b)$, one checks $S_1(a,b) = S_1(a,f)$. Let $g, h \in S_1(a,f)$, we have $gf \leq f$ by $gw^r f$ so that $gh = gfh \leq fh = h$ by the RR-compatibility. Since $S_1(a,f)$ is a rectangular band from Result 6, we have $gh = hgh = h$ which proves that $S_1(a,b)$ is in fact a right zero semigroup.

(2) $\Rightarrow$ (3). Let $e \in E(S)$ and $a \in Reg(eSe)$. Observe that any idempotent in the $\mathcal{L}$-class of eSe containing a is of the form $a'\ a$, where $a' \in V(a) \cap eSe$. Taking $a', a'' \in V(a) \cap eSe$, we have $a'a \in S_1(a'a, e), a''a \in S_1(a''a; e)$. But $S_1(a'a, e) = S_1(a''a, e)$ by $a'a\mathcal{L}a''a$. Thus, we have $a'a = a''aa'a = a''a$, for $S_1(a'a, e)$ is a right zero semigroup.

(3) $\Rightarrow$ (4). Let $a, b \in RegS$ with $a \leq b$ and $c \in S$ with $ac \in RegS$. We proceed to prove $ac \leq bc$.

From Result 2 there are $a' \in V(a), b' \in V(b)$ such that $a'a\omega b'b$ and $a = ba'a$. By $ac \in RegS$, there is $h \in S_1(a,c) = S_1(a'a, c)$ such that $a'ah\mathcal{L}h$. However, we have $h\omega' b'b$ from $a \leq b$. So $b'bh\mathcal{L}h$ which implies $b'bh\mathcal{L}a'ah$. One checks that $b'bh$ and $a'ah$ are $\mathcal{L}$-related in the subsemigroup $b'bSb'b$ which is $\mathcal{L}$-unipotent. Therfore, $b'bh = a'ah$ holds.

On the other hand, there is $x \in S^1$ such that $h = cx$ by $h\omega^\tau c$, we have $ac = ahc = ba'ahc = bb'bhc = bhc = bcxc$. This shows $ac\omega^\tau bc$. Since $a \leq b$, there is $y \in S^1$ such that $a = yb = ya$ which imply $ac = ybc = yac$. Thus, we have $ac \leq bc$.

PROPOSITION 21. *The following five statements concerning a semigroup S with $E(S) \neq \phi$ satisfy the implications:*

$$(1) \Rightarrow (2) \Longleftrightarrow (3) \Longleftrightarrow (4) \Rightarrow (5).$$

(1) $\leq$ is RR-compatible.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(2) For each $e \in E(S), E(eSe)$ is a right regular band.

(3) For any $e, f \in E(S), E(eS)$ is a right regular band and $E(Sf)$ is a right quasi-normal band.

(4) For any $e, f \in E(S)$, whenever $E(eSf) \neq \phi$ it is a right regular band.

(5) $\leq$ is WSRR-compatible.

Proof. (1) $\Rightarrow$ (2). For any $f, g \in E(eSe)$, we have $fg \leq eg = g \leq e$ by $f \leq e$ and the RR-compatibility. So $fg \in E(eSe)$ which shows that $E(eSe)$ is a band. Further, we have $fgf \cdot gf = f(gf)^2 = f(gf), gf \cdot fgf = (gf)^2 = gf$ which implies that fgf and gf are $\mathcal{L}$-related in eSe. But eSe is $\mathcal{L}$-unipotent by Proposition 20, we have $fgf = gf$, that is, $E(eSe)$ is a right regular band.

(2) $\Longleftrightarrow$ (3). For any $f', g \in E(eS)$, because $E(eSe)$ is a right regular band, we have $gf'g = eg \cdot ef' \cdot eg = ege \cdot ef'e \cdot ege \cdot g = ef'e \cdot ege \cdot g = f'g$. So $E(eS)$ is a right regular band. Similarly, for any $g, h, k \in E(Sf)$, we have $ghkh = g \cdot fhf \cdot fkf \cdot fhf \cdot h = g \cdot fkh \cdot fhf \cdot h = gkh$ which shows $E(Sf)$ is a right quasi-normal band. Conversely, because the variety of right regular bands is a subvariety of the that of right quasi-normal bands (Petrich 1977) and $E(eSe) = E(eS) \cap E(Se), E(eSe)$ is a right regular band.

(2) $\Longleftrightarrow$ (4).(4) $\Rightarrow$ (2) is obvious. (2) $\Rightarrow$ (4) can be proved in the same way as that of (2) $\Rightarrow$ (3).

(2) $\Rightarrow$ (5) is immediate from Proposition 21.

COROLLARY 22. (Blyth and Gomes 1983, Theorem 1) For a regular semigroup S, the following five statements are equivalent

(1) $\leq$ is right compatible w.r.t. the operation on S;

(2) For any $e, f \in E(S), S(e, f)$ is a right zero semigroup;

(3) S is locally $\mathcal{L}$-unipotent.

(4) For any $e, f \in E(S), E(eS)$ is a right regular band and $E(Sf)$ is a right quasi-normal band.

(5) For any $e, f \in E(S), E(eSf)$ is a right regular band.

PROPOSITION 23. The following six statements concerning a semigroup S with $RegS \neq \phi$ satisfy the implications:

$$(1) \Rightarrow (2) \Rightarrow (3) \Longleftrightarrow (4) \Longleftrightarrow (5) \Rightarrow (6).$$

(1) $\leq$ is R-compatible;

(2) For any $e, f \in E(S), |S_1(e, f)| \leq 1$;

(3) S is locally subinverse;

(4) For any $e, f \in E(S), E(eSf)$ is a semilattice if it is unempty;

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(5) For any $e \in E(S), E(eS)$ is a right normal band and $E(Se)$ is a left normal band;

(6) $\leq$ is WSR-compatible.

Proof. This proposition is just an easy consequence of the Propositions 21, 22, their duals and the following facts (Petrich 1977, II.3).

(1) The intersection of the varieties of left and right zero semigroups is the variety of the trivial semigroup.

(2) The intersection of the varieties of left and right regular bands is the variety of semilattices.

(3) The intersection of the varieties of right [left] regular and left [right] quasi-normal bands is the variety of right [left] normal bands.

COROLLARY 24. *If Reg S is an ideal of S, then $\leq$ is SR-Compatible if and only if Reg S is a locally inverse semigroup. In particular, $\leq_S$ of a regular semigroup S is compatible w.r.t. the operation on S if and only if S is a locally inverse semigroup.*

The compatibilities defined above clearly satisfy the following implications:

Compatibility $\Rightarrow$ SR-compatibility $\Rightarrow$ R-compatibility $\Rightarrow$ S is a locally subinverse $\Rightarrow$ WSR-compatibility.

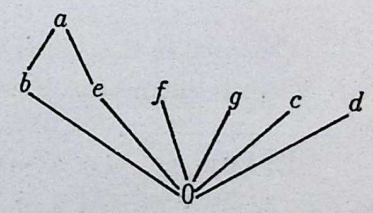
In what follows we give some examples to show that each arrow above may not be conversed. For this end, taking $X = \{1, 2, 3, 4\}$, the full transformation semigroup on X is denoted by $T(X)$ in which every transformation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ i & j & k & l \end{pmatrix}$$

is simplified as $(ijkl)$.

EXAMPLE 3. Let S be the subsemigroup of $T(X)$ generated by $a = (1223), g = (4224)$ and $f = (3232)$. Denoting $e = a^2, c = a^2 f, b = af, d = ag$ and $0 = fa$ we have the multiplication table and the NPO diagram as follows:

	a	e	b	g	f	c	d	0
a	e	e	c	d	b	c	d	0
e	e	e	c	d	c	c	d	0
b	0	0	0	0	b	0	0	0
g	b	0	b	g	0	0	0	0
f	0	0	0	0	f	0	0	0
c	0	0	0	0	c	0	0	0
d	c	0	c	d	0	0	0	0
0	0	0	0	0	0	0	0	0



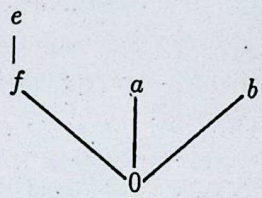
CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

One verifies that $RegS = \{e, f, g, 0\} = E(S)$ and $\leq \cap (RegS \times RegS) = \{0\} \times E(S)$. Thus, $\leq$ is SR-compatible. However, $\leq$ is not compatible because we have $b \leq a, c = eb, e = ea$, but we have no $c \leq e$.

EXAMPLE 4. Let S be the subsemigroup of $T(X)$ generated by $e = (1224), f = (2224)$ and $a = (3223)$. Writing $b = fa, 0 = ae$, we have the multiplication table and the NPO diagram of S as follows:

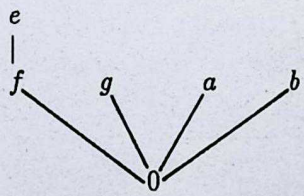
	e	f	a	b	0
e	e	f	a	b	0
f	f	f	b	b	0
a	0	0	0	0	0
b	0	0	0	0	0
0	0	0	0	0	0



One verifies that $RegS = \{e, f, 0\} = E(S)$ which is a chain of length of 3. Thus, $\leq$ is R-compatible. However, it is not SR-compatible, because we have $f \leq e, b = fa, ea = a$ but no $b \leq a$. Further, it is not difficult to check that $\leq$ is SRL-compatible which implies that SRL and SRR-compatibilities are different from each other.

EXAMPLE 5. Let S be the subsemigroup of $T(X)$ generated by $e = (1224), f = (2224)$ and $g = (3233)$. Writing $a = eg, b = fg$ and $0 = ge$, we have the multiplication table and the NPO on S as follows:

	e	f	g	a	b	0
e	e	f	a	a	b	0
f	f	f	b	b	b	0
g	0	0	g	0	0	0
a	0	0	a	0	0	0
b	0	0	b	0	0	0
0	0	0	0	0	0	0



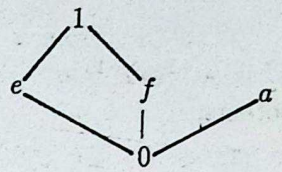
One verifies that $E(S) = \{e, f, g, 0\}$ and $E(xSx)$ is a semilattice for each $x \in E(S)$, i.e., S is locally subinverse. However, we have $f \leq e, b = fg$, and $a = eg$ but no $b \leq a$.

EXAMPLE 6. Let S be the subsemigroup of $T(X)$ generated by $1 = (1234), e = (1224)$ and $f = (3234)$. Writing $a = ef, 0 = fe$, we have the multiplication table and the NPO diagram on S as follows:

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

	1	e	f	a	0
1	1	e	f	a	0
e	e	e	a	a	0
f	f	0	f	0	0
a	a	0	a	0	0
0	0	0	0	0	0



One verifies that $\leq$ is WSR-compatible. However, S is not locally subinverse, since $E(1S1) = \{1, e, f, 0\}$ is not a semilattice.

5. NEI-Semigroups. In this section, we discuss a class of nonregular semigroups called NEI-semigroups on which the NPO is compatible w.r.t. the operation. Using this compatibility we describe the 0-restricted smallest primitive congruence on those members in this class which are categorical at zero. The results obtained in this section extend the corresponding those on inverse semigroups. (Clifford and Preston 1961).

A semigroup S is called an NEI-semigroup, if S is an nilpotent ideal extension of an inverse semigroup.

The class of NEI-semigroups is a proper extension of inverse semigroups, for instance, every finite cyclic semigroup is in this class. It naturally appears in many areas of mathematics and theoretical computer science.

LEMMA 25. *Let S be an* NEI-*semigroup. We have*

(1) *$RegS$ is an ideal and an inverse semigroup itself;*

(2) $\leq \cap (N \times N) = 1_N$ *where* $N = S - RegS$;

(3) *Every homomorphic image of S is also an* NEI-*semigroup.*

Obviously in an NEI-semigroup S every regular element $a \in RegS$ has only one inverse which is denoted by a^{-1}.

PROPOSITION 26. *Let S be a semigroup with $E(S) \neq \phi$. If every $a \in RegS$ has only one inverse denoted by a^{-1}, then the following conditions concerning $a \in RegS$ and any $b \in S$ are equivalent:*

(1) $a \leq b$;

(2) $a = eb$, *for some* $e \in E(S)$;

(3) $a = aa^{-1}b$;

(4) $aa^{-1} = ba^{-1}$;

(5) $a = ba^{-1}a$;

(6) $a = bf$, *for some* $f \in E(S)$;

(7) $a^{-1}a = a^{-1}b$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. It suffices to prove (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (7) $\Rightarrow$ (1), since others can be easily obtained.

(2) $\Rightarrow$ (3). By $a = eb$ we have $aa^{-1}\mathcal{R}a\omega^{\tau}e$ so $aa^{-1}\mathcal{R}aa^{-1}e$. Since aa^{-1} has only one inverse, we have $aa^{-1} = aa^{-1}e$. Thus, $a = aa^{-1}a = aa^{-1}eb = aa^{-1}b$.

(3) $\Rightarrow$ (4). From $a = aa^{-1}b$ we have $ba^{-1} \cdot aa^{-1} = ba^{-1}, aa^{-1} \cdot ba^{-1} = aa^{-1}$ and $(ba^{-1})^2 = ba^{-1}(aa^{-1}b)a^{-1} = ba^{-1}aa^{-1} = ba^{-1}$. So $aa^{-1}\mathcal{L}ba^{-1}$ and $aa^{-1} = ba^{-1}$, being both idempotents.

(7) $\Rightarrow$ (1). By $a^{-1}a = a^{-1}b$ we have $a = aa^{-1}b$. From (3) $\Rightarrow$ (4) $\Rightarrow$ (5) we have $a = ba^{-1}a$ which implies $a \leq b$ by Results 2.

PROPOSITION 27. *If S is an* NEI-*semigroup, then the NPO $\leq$ on S is compatible with respect to the multiplication on S.*

Proof. For any $a, b \in S$ with $a \leq b$, if $a \in S - RegS$, then $b \in S - RegS$ by the Result 2. From Lemma 25 we have $a = b$ and there is no need to prove. Now let $a \in RegS$. From Proposition 26 we have $a = eb$ for some $e \in E(S)$. For any $c \in S$, we have $ac \in RegS$ and $ac = ebc$. So $ac \leq bc$ by Proposition 26 again. Dually, we have $ca \leq cb$.

Writing $S = S^{\circ}$ represents that S is a semigroup with zero 0. A congruence ρ on S is called 0-restricted [primitive], if $0\rho = \{0\}[S/\rho$ is a primitive semigroup]. S is said to be categorical at 0, if for any $a, b, c \in S, ab \neq 0$ and $bc \neq 0$ together imply $abc \neq 0$.

LEMMA 28. *Let $S = S^{\circ}$ be an* NEI-*semigroup which is categorical at* 0. *For any $e, f \in E(S)$ and $x \in S, ex \neq 0$ and $fx \neq 0$ together imply $efx \neq 0$.*

Proof. Since $RegS$ is an ideal of S, we have $f_x \in RegS - \{0\}$ so that there is $h \in S_1(f, x)$ by Result 6. Since $RegS$ is an inverse semigroup and $fh\mathcal{L}h\mathcal{R}hx, fh \in E(S)$, there is a unique $x' \in V(hx) \cap L_h$ such that $xx' = h\omega f$. Clearly $h \neq 0$. From $ex \neq 0, xx' = h \neq 0$ we have $eh = exx' \neq 0$. If $efx = 0$ we should have $eh = efh = efxx' = 0x' = 0$ which is impossible. Thus, $efx \neq 0$.

PROPOSITION 29. *Let S be an ENI-semigroup which is categorical at* 0. *The following binary relation on S.*

$$\beta(S) = \{(x, y) \in S \times S : \exists 0 \neq z \in S, z \leq x, z \leq y\} \cup \{(0, 0)\}$$

is the 0-*restricted smallest primitive congruence on S.*

Proof. $\beta(S)$ is obviously reflexive and symmetric. Let $(x, y), (y, z) \in \beta(S)$. By definition, $\{x, y, z\} \cap \{0\} \neq \phi$ implies $x = y = z = 0$ and the result holds. Now suppose that none of x, y, z is 0. Then there are $u, v \in S - \{0\}$ such that $u \leq x, u \leq y; v \leq y, v \leq z$. If one of u, v is not regular, we have $x = y$ or $y = z$. There is no need to prove. So assume $u, v \in RegS$. From Proposition 26 there are $e, f \in E(S)$ such that $0 \neq u = ex = ey, 0 \neq v = fy = fz$. From Lemma 28 we have $0 \neq efx = fex = fey = efy = efz$ which implies that there is $0 \neq w = efz$ such that $w \leq x, w \leq z$. So $(x, z) \in \beta(S)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

That $\beta(S)$ is a 0-restricted congruence is an immediate consequence of its definition, the compatibility of $\leq$ and that S is categorical at 0.

To prove $\bar{S} = S/\beta(S)$ is primitive, let $\bar{a}, \bar{b} \in \bar{S} - 0$ with $\bar{a} \leq \bar{b}$. If $\bar{a} \in \bar{S} - Reg\bar{S}$, one easily verifies $\bar{a} = \bar{b}$ by Lemma 25. Now suppose $\bar{a} \in Reg\bar{S}$. From Proposition 26 we have $\bar{a} = \bar{e}\bar{b} = \bar{e}b$ for some $e \in E(S)$. Therefore, there exists $c \in S - 0$ such that $c \leq a, c \leq eb \leq b$ which implies $\bar{a} = \bar{b}$.

Now let σ be a 0-restricted primitive congruence on S. For any $(a, b) \in \beta(S) - (0, 0)$, there is $c \in S - 0$ such that $c \leq a, c \leq b$. Because any homomorphism preserves NPO, we have $0\sigma \neq c\sigma \leq_{S/\sigma} a\sigma$ and $0\sigma \neq c\sigma \leq_{S/\sigma} b\sigma$. Since σ is primitive, we have $(a, b) \in \sigma$. Therefore, $\beta(S) \subseteq \sigma$.

PROPOSITION 30. *An* NEI-*semigroup S is primitive if and only if S is a 0-direct union of a unipotent semigroup $N = N^\circ$ with a family $\{S_\alpha : \alpha \in Y\}$ of Brandt semigroups.*

Proof. This is a consequence of Proposition 12. Theorem 4.3, Chapter 2 of (Petrich 1984) and the definition of 0-direct union (cf. III. 3 of (Howie 1976)).

It is well-known that an inverse semigroup $S = S^\circ$ has 0-restricted primitive congruence if and only if S is categorical at zero (see Theorem 7.66 of Cifford and Preston 1961). The necessity part of this result is generally not true of NEI-semigroup. For instance, let S be a 0-direct union of a cyclic nilpotent semigroup $A =< a : a^3 = 0 >$ and a Brandt semigroup B. One verifies that S has a 0-restricted primitive congruence 1_s, but S is not categorical at zero. However, we have the following:

PROPOSITION 31. *Let $S = S^\circ$ be an* NEI-*semigroup. If S has 0-restricted primitive congruence, then for any $a, b, c \in RegS, ab \neq 0, bc \neq 0$ together imply $abc \neq 0$.*

Proof. Assume that ρ is a 0-restricted primitive congruence on S. For any $a, b, c \in RegS$ with $ab \neq 0, bc \neq 0$. We have $a\rho b\rho = (ab)\rho \neq 0\rho$ and $b\rho c\rho = (bc)\rho \neq 0\rho$. Because S/ρ is a primitive NEI-semigroup which is a 0-direct union of a nilpotent semigroup and a family of Brandt semigroups by Proposition 30, the two equalities above and the fact $a\rho, b\rho, c\rho \in Reg(S/\rho)$ imply that $a\rho, b\rho$ and $c\rho$ must be in the same Brandt component of S/ρ and the trace product $a\rho * b\rho * c\rho$ exists. From Lemma 9 we have $a\rho\mathcal{R}a\rho * b\rho * c\rho = a\rho b\rho c\rho = (abc)\rho$ so that $(abc)\rho \neq 0\rho$. Therefore, $abc \neq 0$, since ρ is 0-restricted.

Concerning NEI-semigroups without zero, we have the following.

COROLLARY 32. *Let S be an* NEI-*semigroup without zero. Then S has the smallest primitive congruence*

$$\Gamma(S) = \{(x, y) \in S \times S : \exists z \in S, z \leq x, z \leq y\}$$

which is indeed the smallest group congruence on S.

Proof. When S has no zero, S° is clearly categorical at zero. So $\Gamma(S) \cup (0, 0)$ is the 0-restricted smallest primitive congruence on S° by Proposition 29 which implies that $\Gamma(S)$ is the smallest primitive congruence on S.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Now we prove that $\Gamma(S)$ is indeed the smallest group congruence on S. For any $a \in S$ there is a positive integer $n \geq 2$ such that $a^n \in RegS$. So there exists $b \in RegS$ such that $b \leq a$ by Result 6 and Proposition 7. This implies that $S/\Gamma(S)$ is regular. Furthermore we must have $|E(S/\Gamma(S))| = 1$, since $E(S)$ is a semilattice. Therefore, $\Gamma(S)$ is a group congruence on S.

If σ is a group congruence on S, we have $\Gamma(S) \subseteq \sigma$, since σ is a primitive congruence, completing the proof.

References

Blyth, T. S. and **Gomes, G. M. S.** (1983) : On the Compatibility of the Natural Partial Order on a Regular Semigroup, *Proc. Roy. Soc. Edinburgh, 94A*, 79.

Clifford, A. H. and **Preston, G. B.** (1961) : The Algebraic Theory of Semigroups, *Math. Surveys* No. 7, *Amer. Math. Soc. Providence, R.I.*, I (1961), II (1967).

Fountain, J. B. (1982) : Abundant Semigroups, *Proc. London Math. Mac.* (3) 44, 103.

Howie, J. M. (1976) : *An Introduction to Semigroup Theory*, Academic Press, London, New York.

Mitsch, H. (1986) : A Natural Partial Order for Semigroups, *Proc. Amer. Math. Soc. 97*, 384.

Nambooripad, K. S. S. (1980) : The Natural Partial Order on a Regular Semigroup, *Proc. Edinbergh Math. Soc., 23*, 249.

Petrich, M. (1984) : *Inverse Semigroups*, Jhon Wiley and Sons Inc.

Petrich, M. (1977) : *Lectures in Semigroups*, Akademie Verlag, Berlin.

Tang Xiangdong (1989) : *Generalized Right Adequate Semigroups of Type C*, Thesis for M.Sc Degree in Lanzhou University PRC,

Yu Bingjun (1993) : *The Biorder and Sandwich Sets on a Semigroup with Idempotents, (to appear).*

DEPARTMENT OF APPLIED MATHEMATICS
CHENGDU COLLEGE OF GEOLOGY
CHENGDU, SICHUAN 610059
PRC

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

AUTHOR INDEX

BULLETIN OF THE CALCUTTA MATHEMATICAL SOCIETY

VOLUME 86 (1994)

	Pages
Abdel-All, N. H. : Quasi-Hyperbolic Manifolds	491
Adhikari, M. R. : An Application of Yoneda's Lemma in Semirings	15
Adhikari, M. R. and Das, M. K. : Structure Spaces of Semirings :	313
Ahuja Gopi : On The Product Of An Arbitrary Function And The Fractional Derivative Of Dirac-Delta Distribution-II	463
Al-Ali, M. I. M. and Nhu, Nguyen To : Roberts Example And The AR-Property of Convex Sets In Non-Locally Convex Linear Metric Spaces	419
Al-Baali, M. : A Survey of Classes of Self-Scaling Variable Metric Algorithms	509
Altindis, Huseyin : On P^2_{2j} Sets	305
Bagewadi, C. S. and Siddabasappa : Study of Variably Inclined Rotating MHD Flows In Magnetograph Plane	93
Bandyopadhyay, Rupa and Baisnab, A. P. : Semi-Closure in a Bitopological Space	555
Beltagy, M. : On the Geometry of Specific Types of Points	193
Bhambri, S. K., Kumar, Rajesh and Kumar, Pratibha : On Fuzzy Primary Submodules	445
Bhattacharjya, M. and Rath, J. : A Generalisation of The Generating Functions of Boas and Buck	387
Bhattacharjya, M. and Rath, J. : A Unified class of Orthogonal Polynomials	153
Bhattacharya, D. K. and Manna, S. : On the Space of Entire Dirichlet Series	21
Bilgin, Tunay : The Sequence Space $l(p, f, q, s)$ on Seminormed Spaces**	295
Bingjun, Yu : The Biorder And Sandwish Sets On A Semigroup With Idempotents	453
Bingjun, Yu : The Natural Partial Order on a Semigroup	565
Cakalli, H. and Canak, G. : s—Absolute Almost Convergent Sequence II	379
Chakraborty, Gargi : Existence Theorems, Uniqueness for General Formulation of the Minimum Effort Problems in Locally Convex Linear Topological Spaces	237
Chakraborty, Gargi : On Certain Uniqueness Concepts In Locally Convex Linear Topological Spaces	133
Chakravarty (Hazra), Ratna and Basu, D. K. : An Extension of a Bilateral Generating Function of Modified Jacobi Polynomial	233
Chattopadhyay, Chandan and Bandyopadhyay, Chhanda : C_i-Open Coverings And C_i-Continuous Functions	139

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Pages

Cheng, Shih-Chuan and Mordeson, John N. : Fuzzy Linear Operators and Fuzzy Normed Linear Spaces 429

Choudhury, Binayak S. : Some Results in Fixed Point Theory 47

Constantin, Adrian : On the Approximation of Fixed Points of Operators 323

Cueva, Miguel Caldas : Further Results on Non-Archimedean σ Barrelled in Locally Convex Spaces 249

Das, M. K. and Adhikari, M. R. : Local Homology and Homological Local Connectedness 327

Das, N. R. and Chowdhury, Ajanta : On Core of a Vector Valued Sequence 27

De, U. C. and Ghosh, J. C. : On Pseudo W_2-Symmetric Manifolds 521

De, U. C. and Kamilya, D. : On Generalised Concircular Recurrent Manifolds 69

Dey, S. and Mahato, P. : Effective Elastic Coefficients in an Initially Stressed Medium and Propagation or Rayleigh Waves in such a Half Space Bounded by a Rigid Layer 537

Dey, S. and Mahato, P. : Effective Elastic Constants In a Medium under Initial Stress And Under Gravity And Prediction of Liquid Core Inside A Gravitating Medium 107

Dhage, B. C. : On Continuity of Mappings in D-Metric Spaces 503

Dutta, T. K. and Adhikari, N. C, : On Prime Radical Of Γ-Semigroup 437

Dutta, T. K. and Adhikari, N. C. : Embedding a Γ-Semigroup in a Semigroup with zero 1

El-Sherbiny, H. M. and Abdel-Razek, M. A. : On the Well-Posedness of a Dissipative RLW Class of Equations 481

Elumalai, S. and Ravi, R. : On Remotal Points of Pairs of Sets in Linear 2-Normed Spaces 63

Fengyou, Li : A Fixed Point Theorem for Increasing Operators in Partially Ordered Spaces 59

Ganguly, Ashok and Wadhawa, Kamal : On a Recent Result of Kyfan and its Application to Variational Inequality and Complementary Problems 291

Ganguly, S. and Bandyopadhyay, T. : Further Study of θ–Mods 73

Ghosh, N. K. : On Axisymmetric Waves in Water with an Inertial Surface 373

Hazra (Chakraborty), Ratna and Basu, D. K. : A General Class of Generating Functions of Bessel Polynomials 369

Imdad, M. Ahamad, Aqeel and Khan A. Rauf : On Common Fixed Point of Asymptotically Commuting Mappings 307

Jain, P. K. and Kaushik, S. K. : On K-Shrinking And K-Boundary Complete M-Bases In Subspaces And Superspaces of Banach Spaces 167

John, L. and Rajan, A. R. : Semigroup of Operators with a Parameter 335

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Pages

Kahn, Shabnam; Shukla, D. P. and Bawa, N. P. S. : On Computation of Triangulation with Continuous Refinement of Grid Size 279

Kundu, Krishna : A Plane Vertical Wave Maker Problem in a Two-Fluid Medium Separated by an Internal Surface 7

Mahmood, R. M. S. : A Remark On HNN Groups 115

Mandal, B. N. and Charabarti, A. : On the Green's Function Method For the Sommerfeld Half-Plane Diffraction Problem 145

Mashhour, A. S., Zeyada, F. M. and Ismail, A. H. : On Good Extensions of Some Topological Covering Properties 469

Mittal, M. L. and Kumar, Rajesh : On a Sequence of Fourier Coefficients 349

Miyamoto, Ikuko : Inclusion Theorems for Absolute Nörlund Summability 527

Mohanty, R. and Ray, B. K. : On the Convergence and Absolute Convergence of Some Series Associated with Fourier Series 89

Mukherjee, M. C. and Chongdar, A. K. : A Class of Bilateral Generating Functions of Modified Bessel Polynomials 477

Naggar, A. M. El; Abd-Alla, A. M. and Ahmed, S. M. : Rayleigh Waves in a Magnetoelastic Initially Stressed Conducting Medium with the Gravity Field 243

Naggar, A. M. El; Abd-Alla, A. M. and Ahmed, S. M. : The Stresses in a Rotation of a Non-homogeneous Composite Cylinder of Isotropic Material 205

Narayanan, K. S. and Ramaswamy, V. : Gelfand-Mazur Theorem For Alternative Banach Triple Systems 177

Ojha, Ram Hit and Prasad, Bhagwat : On a Semi-Symmetric Non-Metric Connexion in Almost Grayan Manifolds 213

Padmavati : Approximation of Mean Strong Uniform Convergence of Vector Valued Fourier Series 357

Pandeya, B. M. : Small M-Projective Modules and Submodules of Pseudo Projective Module 497

Parhi, N. and Das, P. : Oscillatoy and Asymptotic Behaviour of a Class of Non-Linear Functional Differential Equations of Third Order 253

Patel, J. : Subclasses of Close-To-Star Functions 361

Patil, P. A. and Chowdhary, M. S. : On Unsymmetric Generalised Operators Of Fractional Integration 119

Prajneshu : A Stochastic Lotka-Volterra Prey-Predator Model with Switching Effect 39

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Pages

Prasad, A. V. M. and Sastri, V. V. S. : An Asymptotic Formula for Partitions into Square Full Integers 33

Prasad, Shanteshwar and Ojha, Ram Hit : On M-Manifolds 413

Ragab, Shahwar F. : Infinite Series Involving Bessel Functions of the same orders 201

Rao, K. V. V. Seshagiri : Non-Oscillation for a Class of Forced Second Order Non-Linear Differential Equations with Damping 275

Sagel, M. Kemal and Akin Omer : An Interpretation of Subharmonic Functions using the Gauss Curvature 319

Sallam, M. H. : Generalised Simple Eigenvalues and Bifurcation for a Linked Multiparameter Eigenvalue Problem 227

Sen, S. K. and Mukherjee, M. N. : θ-Perfect Functions Between Bitopological Spaces 395

Shahna, Abdulla S. Bin : Fuzzy Connectedness and Weaker Forms of Fuzzy Continuity 271

Singh, A. K. and Negi, U. S. : On Projective Motion in a Kaehlerian Space with Recurrent Curvature 531

Singh, A. K. and Petwal, K. C. : On Decomposition of Curvature Tensor Fields in Tachibana Recurrent Spaces 221

Singh, Anand Prakash and Dhar, Raj Shree : On Differential Polynomials 171

Sinha, B. B. and Yadav, R. N. : Semi-Invariant Submanifolds Of a Kenmotsu Manifold 405

Zaki, Adel M. : The Pure State Spaces of Certain C*-Algebras 267

Zayed, E. M. E. and Ibrahim, S. F. M. : Eigenfunction Expansions for a Regular Right-Definite Eigenvalue Problem of Order 2N with an Eigenparameter in the Boundary Conditions 547

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

THE COUNCIL OF THE
CALCUTTA MATHEMATICAL SOCIETY
1994–1996

President

M. Dutta

Vice-Presidents

T. K. Mukherjee, P.P. Chatterjee, L. Debnath
S. P. Pal, K. R. Ramachandran

Treasurer

U. Basu

Secretary

R. N. Sen

Editorial Secretary

C. G. Chakrabarti

Assistant Secretary

Sulekha Mukherjee

Other Members of the Council

B. C. Chakraborty, N. C. Ghosh, A. B. Roy, N. D. Sengupta,
M. R. Adhikari, P. K. Chaudhuri, D. K. Basu, M. Tarafdar,
P. K. Das, B. K. Sengupta, R. N. Chakrabarti, L. M. Saha

All correspondence with the society, (subscriptions to the Bulletin, admission fees and annual contributions of members etc.) are to be made with :

The Secretary

Calcutta Mathematical Society

'Asutosh Bhavan', AE–374, Sector–I, Calcutta 700 064 (INDIA)

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CALCUTTA MATHEMATICAL SOCIETY

AE 374, SECTOR I, SALT LAKE CITY, CALCUTTA 700 064, INDIA

NOTES FOR AUTHORS

On General Requirements

1. The Bulletin of the Calcutta Mathematical Society publishes papers presented to the societ containing original works in different branches of mathematics and allied sciences.

2. Authors should send to the *Editorial Secretary* of the Society two copies of their communications (one original copy) typed neatly on white paper, on one side only, with double spacing and with a margin of cm. together with all ancillary materials, while a further copy should be retained by them.

3. Authors will receive ordinarily one proof for corrections. Corrections to the proofs should b restricted to printer's errors only and made according to standard practice. Cost of any other corrections wi be charged to the authors.

4. Authors are required to inform the Editorial Secretary of any change of address which occur whilst their papers are in the process of publication.

5. The printing cost of the paper at the rate of Rs. 60.00 or $ 20 per page should be paid by th Authors or their Institutions.

LIBRARY
Gurukul Kangri Vishwavidyalaya
HARIDWAR
130271

On Script Requirements

1. The first page should bear in order—

The title, brief but informative; the name(s) of the author(s); the address to which the proofs shoul be sent; a running title of not more than 55 letters or signs; a brief abstract followed by (i) Major field(s (ii) AMS Subject classification and the introduction.

2. Equations and symbols may be handwritten or typewritten; clarity for the printer is essentia Unusual symbols should be identified by name in the margin (in pencil) the first time they are used. If hand written, care should be taken to distinguish between capital and lower case letters *e.g.* c, k, o, p, s, w, phi, ps between alpha (α) and a; kappa (κ) and k; mu (μ) and u; nu (ν) and v; eta (η) and n; omega (ω) and When used as order symbol o and 0 should be shown as $\underline{o}$ and $\underline{\underline{O}}$.

Fractional exponents are to be used instead of root sign, n ! for $\lfloor n$ exp( ) notation when the exponen is complicated. Repetition of complicated expression is to be avoided by representing it with a symbol.

3. Reference to the literature in the text should be by author's surname and year in parenthese References cited in the text should be listed at the end of the paper in the alphabetical order of the surnam of the author in the following way (a recent number of the Bulletin may be a proper guide):

Chandrasekhar, S. (1961) Hydrodynamic and Hydromagnetic Stability, Oxford University Press, London, p. 10.

Wilson, B. M. (1922) Proc. Lond. Math, Soc. (2), 21, 235.

4. Tables should be typed on pages separate from the text, numbered sequentially and planned to f the page width of the journal (13 cm.). Table numbers should be typed separately in sequence on the manuscrip paper.

5. Line drawing must be clearly and sharply drawn (duly traced by a draughtsman) with black Ind ink on white tracing paper. Figures should be drawn about 4 times the size which would appear in prin Standard size after reduction is normally 7 cm. by 8 cm. Each figure must have a number. Figure numbe should be typed in sequences separately on manuscript paper. The back of each figure should bear the name(of author(s) and the figure number.

Entered in Database
1/02/04
Signature with Date

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar